KV-Topics
4/10/25

Moving On: The Zimmer Conjecture and Related Problems

"Short version": G semisimple, $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$, $\Gamma \leq G$ lattice

$\Rightarrow$ Γ can't act on an n-manifold unless $\exists$ algebraic action on an n-manifold.

- Let G be a semisimple group,

$$n(G) = \min\left\{ n \;\middle|\; \exists \rho : G \to GL(n,\mathbb{R}),\ \rho \not\equiv e \right\}$$

$$v(G) = \min\left\{ \dim(G/H) \;\middle|\; H \subset G \right\}.$$

$r(G) =$ minimal resonant codimension

$\hookrightarrow$

$$\Delta = \text{roots of } G$$

$$\hat{\Delta} = \Delta / \mathbb{R}_+ = \left\{ [\alpha] \;\middle|\; \begin{array}{l} \beta \in [\alpha] \\ \Leftrightarrow \beta = \lambda\alpha, \\ \exists \lambda \in \mathbb{R}_{>0} \end{array} \right\}$$

$\mathfrak{q}$ = parabolic subalgebra of $\mathfrak{g}$

$$r_G(\mathfrak{q}) = \#\left\{ [\alpha] \in \hat{\Delta} \;\middle|\; \mathfrak{g}_{[\alpha]} \not\subset \mathfrak{q} \right\}$$

$$r(G) = \min \left\{ r_G(\mathfrak{q}) \;\middle|\; \begin{array}{c} \mathfrak{q} \subset \mathfrak{g} \\ \text{parabolic} \end{array} \right\}$$

- G_{compact} = compact real form of G.

<u>Recall</u>: For $\mathfrak{g}$ real s.s. Lie alg

$$\begin{tikzcd} & \mathfrak{g} \arrow[d, squiggly] & \\ \text{split real form} & \mathfrak{g}_{\mathbb{C}} \arrow[l] \arrow[r] & \text{compact real form} \end{tikzcd}$$

$$\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}) \qquad \operatorname{rank}_{\mathbb{R}}(\mathfrak{g}_{\text{compact}}) = 0$$

- Two types of G-actions

vol.-preserving (affine on nilmanifolds)

non-vol.-preserving (boundary on G/Q).

- <u>Zimmer's Conjectures</u>: Assume $\Gamma \curvearrowright M$ by C^∞ diffeos. (M compact)
  - If $\dim(M) < \min\{v(G_{compact}), v(G)\}$, then Γ factors through a finite group.
  - If $\dim(M) < \min\{v(G_{compact}), n(G)\}$ and $\Gamma \curvearrowright M$ preserves volume, then Γ factors ~~through~~ through a finite group

  ↳

- If $\dim(M) < n(G)$ and $\Gamma \curvearrowright M$ preserves a volume, then the action preserves a (smooth) Riemannian metric.
- If $\dim(M) < v(G)$, then the action preserves a (smooth) Riemannian metric.

<u>Theorem (Brown-Fisher-Hurtado)</u>:

The conjecture(s) hold for $\mathbb{R}$-split groups

<u>Ex</u>: $G = SL(d, \mathbb{R})$.

$$n(G) = d$$
$$v(G) = d-1$$
$$r(G) = d-1.$$

Rem: (Symmetric spaces)

$G =$ s.s. Lie group

Cartan involution

$\theta : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$

$\mathrm{Fix}(\theta) = \mathfrak{k}$ (compact)

-1 space $= \mathfrak{p}$ (symmetric matrices)

Ex: $SL(d,\mathbb{R})$

$\theta : A \mapsto -A^T$

$\Gamma \backslash SL(d,\mathbb{R}) / SO(d)$

Thm (Brown - Rodriguez Hertz - Wang):

1) Assume $\Gamma \subset SL(d,\mathbb{R})$ is a uniform lattice and $\Gamma \curvearrowright M^{d-1}$. Then either

- Γ preserves a smooth volume, or
- Γ is measurably (smooth?) conjugate to the boundary action on $\mathbb{RP}^{d-1}$.

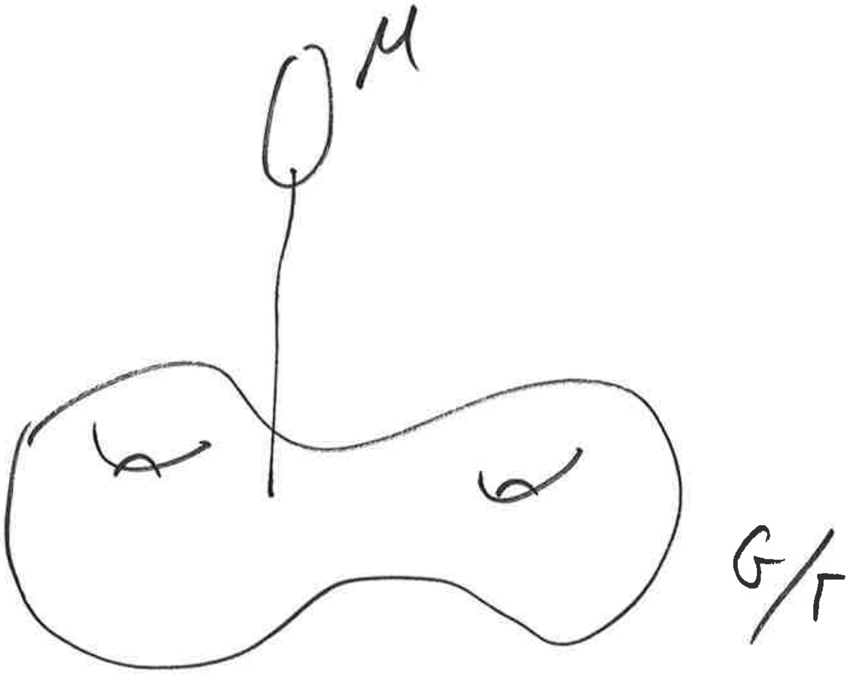
2) Assume $\Gamma \subset G$ is a uniform lattice and $\Gamma \curvearrowright M^{r(G)}$. Then either

- Γ preserves a Borel prob., or
- $M = G/Q$, $G \curvearrowright M$ by translations

(regularity? mble)

Tool: Resonances.

suspend Γ-action : $G \curvearrowright \overbrace{(G \times M^d)}^{=X} / \Gamma^{\Delta}$



In the volume-preserving case,

$\exists$ G-inv. prob. measure μ on X.

Applying Zimmer, $\exists$ rep $\rho : G \to GL(d, \mathbb{R})$

such that $dg|_{TM} \sim \rho \times$ compact

Def: ρ is called nonresonant if every weight of ρ is nonproportional to every root ($=$ weights of Ad).

Lemma / Exr: If ρ does not have 0 as a weight, then it is nonresonant.

(**Trick**: $\rho(\mathfrak{g}_\alpha)\, E_\beta = E_{\beta+\alpha}$.)

"Anosov condition"

Thm (Brown - RH - Wang): If $\Gamma \curvearrowright N/\Lambda$ is an action on a nilmanifold with an Anosov element and the action lifts to N, then it is C^∞ conjugate to an automorphism action.

Thm (Franks–Manning): If $f : N/\Lambda \circlearrowleft$ is Anosov, then f is C^0-conjugate to an automorphism.

Options for the rest of the semester:

- → Brown–Fisher–Hurtado ($<$ crit. dim.)
- → Brown–RH–W; Lee (in crit. dim.)
  - ↪ uniform assumptions.
- → Damj.–Spat.–V-Xu (partially hyperbolic G-actions)