KV-Topics
4/1/25

Back to proof of Zimmer, with more details:

$G \curvearrowright P \curvearrowleft H$

$\downarrow$

$G \curvearrowright (X, \mu)$

$\omega : P \to V$ A-invariant
H-equivariant

$\phi : P \to Rat(G, V)$

$\phi(p)(g) = \omega(g^{-1} p)$

$\boxed{\text{Starting point}}$: ϕ takes values in a single $G \times H$-orbit.

$G \times H \curvearrowright Rat_d(G; V)$

Let $M_p = Stab_G(\phi(p))$ and (degree d rational functions.)

$G_p = Stab_G(H \cdot \phi(p))$, where $G \curvearrowright {}_H\backslash Rat_d(G; V)$

Then M_p and G_p are both algebraic groups.

(These are rational actions.)

Furthermore, $G_p = \pi_G(\mathcal{M}_p)$:

$$(g,h)\cdot \phi(p) = \phi(p)$$

$$\Leftrightarrow \quad g\cdot\phi(p) = h^{-1}\phi(p).$$

<u>Lemma</u>: $G_p = G$ a.e. p.

More precisely, for μ-a.e. $pH \in (X,\mu)$, so $\forall h: G_{ph} = G_p$.

<u>Pf</u>: Define

$$q: X \to \underbrace{\mathrm{Im}}_{\cong G/G_{p_0}}\left(\phi : P/H \to {}_H\backslash \mathrm{Rat}\right)$$

Then $q_*\mu$ is G-invariant on G/G_{p_0}

$\Rightarrow$ $G/G_{p_0} = *$ $\Rightarrow$ $G_{p_0} = G$.

(by Rosenlicht)

$\Rightarrow$ ϕ takes values in a single H-orbit.

Let $K_p \subseteq H$ be $K_p = \{ h \in H \mid h\,\phi(p) = \phi(p) \}$

$$= \mathrm{Stab}_H(\phi(p)).$$

$$\begin{aligned}
h \in K_{gp} &\Leftrightarrow h\,\phi(gp) = \phi(gp) \\
&\Leftrightarrow (g^{-1}h)\,\phi(p) = g^{-1}\,\phi(p) \\
&\Leftrightarrow h\,\phi(p) = \phi(p) \\
&\Leftrightarrow h \in K_p
\end{aligned}$$

Furthermore,

$$K_{ph} = h\,K_p\,h^{-1}, \quad \text{as}$$

$$\begin{aligned}
k \in K_{ph} &\Leftrightarrow k\,\phi(ph) = \phi(ph) \\
&\Leftrightarrow k h^{-1}\,\phi(p) = h^{-1}\,\phi(p) \\
&\Leftrightarrow h k h^{-1}\,\phi(p) = \phi(p) \\
&\Leftrightarrow h k h^{-1} \in K_p.
\end{aligned}$$

Define

$$Q = \{p \mid K_p = K_{p_0}\}$$

Claim: Q is a $N_H(K_{p_0})$-reduction of P

P is the algebraic hull

$$\Rightarrow N_H(K_{p_0}) = H \Rightarrow K_{p_0} \lhd H.$$

Cor: $K := K_p$ is (x) independent of p

H/K acts freely on $\mathrm{Im}(\phi) \underset{K}{\subseteq} \mathrm{Rat}(G, V)$

Def: Let $\eta_p(g)$ be the unique element of H/K such that $\phi(gp) = \eta_p(g)\phi(p)$.

(By Oseledets, if H is semisimple, $H = (H/K) \times K$ up to finite groups.)

Claim: η_p is an antihomomorphism $\forall p$.

Pf:

Step 1: Cocycle property

Step 2: Upgrade.

(1) $$\phi(g_1 g_2 p) = \eta_p(g_1 g_2)\phi(p)$$

$$\eta_{g_2 p}(g_1)\phi(g_2 p) = \eta_{g_2 p}(g_1)\,\eta_p(g_2)\,\phi(p)$$

$$\Rightarrow \boxed{\eta_p(g_1 g_2) = \eta_{g_2 p}(g_1)\cdot \eta_p(g_2).}$$

↳

② We'll show $\boxed{\eta_{ph}(g) = h^{-1}\eta_p(g)h}$.

$$\phi(gph) = \eta_{ph}(g)\cdot\phi(ph)$$

$$= \eta_{ph}(g)\ h^{-1}\phi(p)$$

$$= h^{-1}\phi(gp)$$

$$= h^{-1}\eta_p(g)\ \phi(p)$$

$$\Rightarrow \eta_{ph}(g)h^{-1} = h^{-1}\eta_p(g)$$

$$\Rightarrow \eta_{ph}(g) = h^{-1}\eta_p(g)h.$$

$$\phi(g_2 p) = \eta_p(g_2)\phi(p) = \phi\left(p\ \eta_p(g_2)^{-1}\right)$$

$$\phi(g g_2 p) = \phi\left(g p\ \eta_p(g_2)^{-1}\right)$$

$$\Rightarrow \eta_{g_2 p}(g) = \eta_{p\,\eta_p(g_2)^{-1}}(g) = \eta_p(g_2)\eta_p(g)\eta_p(g_2)^{-1}$$

$$\hookrightarrow = \cdots$$

$$\cdots = \eta_p(g_2)\,\eta_p(g_1)\,\eta_p(g_2)^{-1}\,\eta_p(g_2)$$

$$= \eta_p(g_2)\cdot\eta_p(g_1).$$