MATH3210 - SPRING 2024 - SECTION 001

HOMEWORK 4

Problem 1 (20 points)**.** Let (c_n) denote any sequence of real numbers such that $0 < c_n < 1$. Show that if (a_n) is a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = c_n \cdot a_n$, then (a_n) converges. [*Hint*: You won't be able to use the definition of a limit directly, as some of these sequences won't converge to 0! What tools do we have for proving a limit exists without knowing what the limit is?]

Problem 2 (40 points)**.** Let (a_n) be a sequence of real numbers such that $a_n > 0$ for every $n \in \mathbb{N}$. For each, prove or find a counterexample:

(a) If a_n diverges to ∞, then $1/a_n \to 0$.
(b) If $1/a_n \to 0$, then (a_n) is unbounded.
(c) If (a_n) is bounded above, then $(1/a_n)$ has a convergent subsequence.
(d) If $\liminf a_n > 0$, then $(1/a_n)$ has a convergent subsequence.

For the next two exercises, we consider the *asymptotic variation* of a sequence (a_n). If (a_n) is a sequence, let $V = \limsup a_n - \liminf a_n$. We assume throughout that $V < \infty$.

Problem 3 (20 points)**.** Show that for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist indices $m, n \in \mathbb{N}$ such that $m, n \geq N$ and $a_m - a_n > V - \varepsilon$.

Problem 4 (20 points)**.** Give an example of a sequence such that for any two indices $m, n \in \mathbb{N}$, $a_m - a_n < V$. Justify your answer (ie, calculate V for your sequence and prove that your calculation is correct. Then verify the given property).

Problem 5 (Ungraded, extra practice)**.** Show thta for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ satisfy $m, n \geq N$, $|a_m - a_n| < V + \varepsilon$.