Problem 1. Show that if \( M_i \) are compact, connected, oriented manifolds of the same dimension for \( i = 1, 2, 3 \), and \( f : M_1 \to M_2 \) and \( g : M_2 \to M_3 \) are \( C^\infty \) functions, then \( \deg(g \circ f) = \deg(g) \deg(f) \).

Problem 2. Show that if \( M \) and \( N \) are compact, connected and oriented manifolds of the same dimension, and \( F : M \to N \) has \( \deg(F) = 0 \), then \( F \) has a critical point.

Problem 3. Let \( A \) be an \( n \times n \) square matrix with integer entries, \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \) and \( F_A : \mathbb{T}^n \to \mathbb{T}^n \) be defined by

\[
F_A([x]) = [Ax]
\]

where the notation \([x]\) denotes the equivalence class \(x + \mathbb{Z}^n\).

1. Show that \( F_A \) is well-defined.
2. Compute \( \deg(F_A) \) by computing the signed number of preimages of a regular value of \( F_A \).
3. Compute \( \deg(F_A) \) by computing \( \int F_A^*\omega \), where \( \omega \) is the standard \( n \)-form \( dx_1 \wedge \cdots \wedge dx_n \).

Non-graded.

Problem 4. Let \( \Gamma \) be a countable group acting properly discontinuously on an oriented manifold \( M \) by diffeomorphisms. Show that the quotient manifold \( M/\Gamma \) is orientable if and only if every \( \gamma \in \Gamma \) is orientation-preserving.

Problem 5. Let \( \omega \) be a top form on a \( C^\infty \) manifold \( M \) and \( \varphi_t : M \to M \) be a flow on \( M \) generated by a vector field \( X \). Show that \( \varphi_t^*\omega = \omega \) for all \( t \in \mathbb{R} \) if and only if \( \iota_X\omega \) is closed. Use this to find a condition for a flow generated by a vector field \( X = \sum f_i \frac{\partial}{\partial x_i} \) to preserve the standard top form \( \omega = dx_1 \wedge \cdots \wedge dx_n \) on \( \mathbb{R}^n \).

Problem 6. Let \( f : M \to M \) be an orientation-preserving diffeomorphism of an oriented compact manifold \( M \). Fix a non-vanishing top form \( \omega_0 \). Show that there is a unique \( C^\infty \) function \( \lambda : M \to \mathbb{R} \) such that \( f^*\omega = e^\lambda \omega \), and \( f \) preserves some non-vanishing top form if and only if there exists a \( C^\infty \) function \( h : M \to \mathbb{R} \) such that

\[
\lambda = h \circ f - h.
\]

Problem 7. If \( G \) is a connected, compact Lie group, consider the squaring map \( s(g) = g^2 \).

(a) Show that if \( G = \mathbb{T}^n \), then \( \deg(s) = 2^n \). Check this formula both by computing the degree at a regular value, as well as using a volume form.

(b) Show that if \( G \) is the set of unit quaternions, then \( \deg(s) = 2 \) [Hint: Show that if \( g \neq \pm 1 \), then there is a unique 1-parameter subgroup passing through \( g \) and that any square root must belong to it]

Remark 1. If you know about the structure of compact Lie groups, you may try the following more difficult exercise: \( \deg(s) = 2^r \), where \( r \) is the maximal dimension of a connected abelian subgroup of \( G \).