Problem 1. Let $G$ be a connected Lie group. Show that $Z(G) = \text{ker}(\text{Ad})$, where $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

Problem 2. Classify the 2-dimensional connected Lie subgroups of Heis = \[
\left\{ \begin{pmatrix}
 1 & x & z \\
 0 & 1 & y \\
 0 & 0 & 1
\end{pmatrix} : x, y, z \in \mathbb{R} \right\}.
\]

Problem 3. Let $G$ be the group of transformations of $\mathbb{R}^2$ obtained by compositions of translations and homotheties $x \mapsto \lambda x$ for $\lambda \in \mathbb{R}_+$.

1. Show that $G$ is center-free (ie, that $Z(G) = \{e\}$).
2. Find vector fields on $\mathbb{R}^2$ generating the actions by homotheties and translations.
3. Show that the vector fields of the previous part form the basis of a Lie algebra (ie, that they span a space closed under Lie brackets).
4. Compute the adjoint representation of its Lie algebra in the basis of the previous part.
5. Build a matrix group $H$ isomorphic to $G$.

Non-Graded.

Problem 4. If $G$ is a Lie group, let $\hat{P}$ denote the set of continuous paths $\gamma : [0, 1] \to G$ such that $\gamma(0) = e$. Define a multiplication

\[(\gamma_1 * \gamma_2)(t) = \begin{cases}
  \gamma_2(2t), & t \in [0,1/2] \\
  \gamma_1(2t-1)\gamma_2(1), & t \in [1/2,1]
\end{cases}
\]

Define a relation that $\gamma_1 \sim \gamma_2$ if and only if $(\gamma_1 * \gamma_2)(1) = e$, and $\gamma_1 \sim \gamma_2$ is trivial in $\pi_1(G,e)$. Show that $\sim$ is an equivalence relation, $*$ descends to $P = \hat{P}/\sim$ as a well-defined group operation, and that the map $\pi : \hat{P} \to G$ defined by $\pi(\gamma) = \gamma(1)$ is a group homomorphism. Furthermore, build a topology on $P$ so that $\pi$ is a covering map. (This shows that the universal cover of a Lie group is a Lie group!)

Problem 5. Find all 2-dimensional connected Lie subgroups of $SL(2,\mathbb{R})$ up to conjugacy.

Problem 6. * Find all 2-dimensional connected Lie subgroups of $SL(3,\mathbb{R})$ up to conjugacy.