1) Let \( F_{x,r} : \mathbb{R}^3 \to \mathbb{R} \) be defined by

\[
F_{x,r}(z) = \|x-z\|^2 - r^2
\]

Note that when \( r > 0 \), 0 is a regular value of \( F_{x,r} \) and \((F_{x,r})^{-1}(0)\) is the sphere of radius \( r \), centered at \( x \). Furthermore, 
\((F_{x,r})^{-1}(0) = (F_{x_1,r_1})^{-1}(0) \) if and only if \((0,0)\) is a regular value of 
\[
\mathcal{G}(z) = (F_{x_1,r_1}(z), F_{x_2,r_2}(z))
\]

Note that 
\[
\begin{pmatrix} 
1 & 2(z^{(1)} - x_1^{(1)}) & 2(z^{(2)} - x_1^{(2)}) & 2(z^{(3)} - x_1^{(3)}) \\
2(z^{(1)} - x_1^{(1)}) & 1 & 2(z^{(2)} - x_1^{(2)}) & 2(z^{(3)} - x_1^{(3)}) \\
2(z^{(2)} - x_1^{(2)}) & 2(z^{(3)} - x_1^{(3)}) & 1 & 2(z^{(3)} - x_1^{(3)}) \\
2(z^{(3)} - x_1^{(3)}) & 2(z^{(3)} - x_1^{(3)}) & 2(z^{(3)} - x_1^{(3)}) & 1 
\end{pmatrix}
\]

This matrix has full rank if and only if its rows are linearly independent.

Hence, the intersection is non-transverse if and only if there exists \( z \in \mathbb{R}^3 \) belonging to the intersection such that

\[
(*) \quad z - x_1 = \lambda (z - x_2)
\]

for some \( \lambda \in \mathbb{R} \). By taking norms, and using that \( \|z - x_1\| = r_2 \), we conclude that \( |\lambda| = r_2 / r_1 \).

Finally, equation (*) implies that \( x_1, x_2, \) and \( z \) lie on the same line. It then follows that non-transversality occurs exactly when either:

\[
\|x_1 - x_2\| = r_1 - r_2 \quad \text{or} \quad \|x_1 - x_2\| = r_1 + r_2
\]
2) Define \( F: \mathbb{R}^2 \to \mathbb{R}^3 \) by

\[
F(x, y) = (x, y, f(x, y))
\]

Then \( F \) is a homeomorphism onto its image, and

\[
dF_{xy} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
f_x(x, y) & f_y(x, y)
\end{pmatrix}
\]

Since \( F \) is an immersion and homeomorphism onto its image, it is an embedding.

The tangent bundle to \( \Gamma_f \) is given by

(i) \( \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ f_x(x, y) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ f_y(x, y) \end{pmatrix} \right\} \)

and

(ii) \( \ker \phi^\perp \) where

\[
\phi_{x,y}(v_1, v_2, v_3) = v_3 - f_y(x, y)v_2 - f_x(x, y)v_1
\]

Finally, if \( f \) and \( g \) are \( C^0 \), note that \( \Gamma_f \) and \( \Gamma_g \) are intersect in \( f^{-1}(g(x, y)) \), where

\[
h(x, y) = g(x, y) - f(x, y).\]

Furthermore, the planes in (i) coincide for \( f \) and \( g \) if and only if

\[
\nabla f = \nabla g \quad \text{or} \quad \forall h = 0.
\]
Hence, we may find a transverse intersection when \( h \) has a regular value. By Sard's Theorem, the set of regular values has full measure, and by definition of \( h \), the image is a nontrivial interval unless \( \gamma_g \) is a constant, hence \( \gamma_g \equiv c \) for some \( c \) unless \( \nabla f \equiv \nabla g \).

3) Consider the function \( F(A, \nu) = A\nu \), so \( F : M_2(\mathbb{R}) \times S^1 \to \mathbb{R}^2 \). We claim that \( F \notin S^1 \). Indeed,

\[
\begin{align*}
&\Lambda F \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \right) = \begin{pmatrix} 2\nu_1 + \beta \nu_2 \\ \alpha \nu_1 + \beta \nu_2 \end{pmatrix} \\
&\text{Therefore,}
\end{align*}
\]

\[
\begin{align*}
&dF \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \right) \wedge \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \\
&\begin{pmatrix} d\nu_1 + a\nu_1 + \beta\nu_2 + b\nu_2 \\ c\nu_1 + \alpha\nu_2 + \beta\nu_2 \end{pmatrix}, \nu_1 \nu_1 + \nu_2 \nu_2 = 0
\end{align*}
\]

Now, if \( F(A, \nu) \in S^1 \), then \( \nu \neq 0 \). Then by equation (\#), \( dF(A, \nu) \) is zero, by choosing \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) appropriately, and taking \( \nu = 0 \). By Sard's Theorem, for almost every \( A, F(A, S^1) \).
To see when the intersection is nontrivial, note that $F_A(v) = Av \in S^1$ if and only if $\|Av\| = 1$. Thus, we require that

$$\|A\| = \sup_{v \in S^1} \|Av\| \geq 1$$

and

$$\inf_{v \in S^1} \|Av\| \leq 1.$$  

To see when $F_A$ is an immersion, note that

$$T_{vS^1} = \mathbb{R} \cdot (-v_2/v_1).$$

Then

$$dF_A(v) (t(-v_2/v_1)) = tA(-v_2/v_1),$$

and this is nonzero at every point if and only if $\ker A = 0$. That is, when $A$ is invertible. Since $(F_A)^{-1} = F_A^{-1}$, it follows that TFAE:

- $F_A$ is an immersion
- $F_A$ is an embedding
- $A$ is invertible.
4) Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), and note that

\[
F(A) = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}
\]

Therefore,

\[
dF(A) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c & a & d & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]