SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 1

In the following problems, a “fake \( n \)-manifold” is a topological space \( M \) which is locally Euclidean. That is, for every point \( p \in M \), there exists a neighborhood \( U \subset M \) of \( p \), an open set \( V \subset \mathbb{R}^n \) and a homeomorphism \( \varphi : U \to V \).

**Problem 1.** Let \( M \) be a connected topological \( n \)-manifold, and \( C \subset M \) be a closed proper subset. Let \( M^f \) be the set

\[
M^f = \left\{(x,a) : x \in M, \text{ and } \begin{cases} a = 0, & x \notin C \\ a = \pm 1, & x \in C \end{cases} \right\}
\]

Define a topology on \( M^f \) as being generated by two types of open sets from derived from the topology on \( M \): If \( U \subset M \) is open, let

- \( U^\# = \{(x,0) : x \in U \setminus C\} \cup \{(x,1) : x \in C \cap U\} \), and
- \( U^\flat = \{(x,0) : x \in U \setminus C\} \cup \{(x,-1) : x \in C \cap U\} \).

Show that, with the topology generated by sets of the form \( U^\# \) and \( U^\flat \), \( M^f \) is a fake \( n \)-manifold, but not a manifold. What happens if \( C = M \)?

**Remark 1.** In the previous problem, when \( M = \mathbb{R} \) and \( C = \{0\} \), this is often called the “line with two origins.”

**Problem 2.** Give an example of a Hausdorff fake \( n \)-manifold which is not a manifold (and justify why it is not a manifold). \([\text{Hint: A disjoint union of locally Euclidean spaces is still locally Euclidean}]\)

**Remark 2.** The example you come up with in the previous problem is probably not connected. For a connected example, look up a pathology called the *long line*.

**Problem 3.** Show that \( S^n = \left\{x \in \mathbb{R}^{n+1} : ||x|| = 1\right\} \) has a canonical smooth \( n \)-manifold structure by explicitly finding a smooth atlas and showing the atlas is smooth.

**Problem 4.** Show that if \( M \) is a smooth \( m \)-manifold and \( N \) is a smooth \( n \)-manifold, then \( M \times N \) has a canonical smooth \((m+n)\)-manifold structure.

**Problem 5.** Show that \( \mathbb{R}^2 \setminus \{0\} \), \( A = \{x \in \mathbb{R}^2 : 1 < ||x|| < 2\} \) and \( S^1 \times (0,1) \) are all diffeomorphic with their standard smooth structures. \([\text{Hint: Find explicit diffeomorphisms between them. Show that the maps you find are bijective and differentiable, with invertible derivative}].\)

**Problem 6.** Let \( X \) denote the boundary of the unit square in \( \mathbb{R}^2 \). Prove or find a counterexample:

1. \( X \) is a topological 1-manifold.
2. There exists a smooth structure on \( X \).
3. There exists a smooth structure on \( X \) such that the inclusion of \( X \) into \( \mathbb{R}^2 \) is \( C^\infty \).
4. There exists a smooth structure on \( X \) such that the inclusion of \( X \) into \( \mathbb{R}^2 \) is an immersion.
Linear algebra and vector calculus review

Problem 7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and assume that there exists a $v \in \mathbb{R}^n$ such that $v$ is an eigenvector of $dF(x)$ with real eigenvalue for every $x \in \mathbb{R}^n$. Show that the lines $L(x) = \{x + tv : t \in \mathbb{R}\}$ are equivariant: $F(L(x)) = L(F(x))$.

Problem 8 (Contraction mapping principle, differentiable version). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that the eigenvalues of $DF(x)$ all have modulus at most some $\lambda < 1$ for every $x \in \mathbb{R}^n$. Show that $F$ has a unique fixed point $x_0$, and for every $x \in \mathbb{R}^n$, $F^k(x) \to x_0$ as $k \to \infty$.

Problem 9. Let $V$ and $W$ be (real) finite-dimensional vector spaces and $\text{Hom}(V,W)$ be the set of linear transformations from $V$ to $W$. 

(1) Show that $\text{Hom}(V,W)$ is a real vector space.
(2) With fixed bases for $V$ and $W$, find an isomorphism between $\text{Hom}(V,W)$ and $M(m,n)$, the set of $m \times n$ matrices, where $m = \dim(V)$ and $n = \dim(W)$.
(3) If $V_0 \subset V$ is a subspace of $V$, let $\text{Ann}(V_0) \subset \text{Hom}(V,W)$ be the annihilator of $V_0$. That is, the set of $\varphi \in \text{Hom}(V,W)$ such that $\varphi(v) = 0$ for all $v \in V_0$. Show that $\text{Ann}(V_0)$ is a vector subspace of $\text{Hom}(V,W)$, then find and prove a formula for $\dim(\text{Ann}(V_0))$ in terms of $\dim(V)$, $\dim(W)$ and $\dim(V_0)$. [Hint: It might be useful to think about it as matrices using the previous part]
(4) * Find a canonical isomorphism between $V^* \otimes W$ and $\text{Hom}(V,W)$, and prove it is an isomorphism. Construct a projection $\pi : V^* \otimes W \to V_0^* \otimes W$ such that $\text{Ann}(V_0) = \ker \pi$, and prove that it is a projection, and that the kernel is as described. Deduce the formula for $\dim(\text{Ann}(V_0))$ using $\pi$, as well.