Complete as many problems as you can. The 3 best solutions will be counted towards your score.

**Problem 1.** Let $G$ and $H$ be connected Lie groups, and $F : G \to H$ be a surjective, $C^\infty$ homomorphism. Show that $F$ is a submersion.

**Problem 2.** Prove or find a counterexample: for every $C^\infty$ connected manifold $M$, there exists a vector bundle $E$ such that $E \oplus TM$ is a trivial bundle over $M$.

**Problem 3.** Give an example of a 2-dimensional distribution on the Lie group $SL(2, \mathbb{R})$ which is not integrable. Justify your claims.

**Problem 4.** Let $\alpha$ be a 1-form on a connected 3-manifold $M$ such that $\alpha \wedge d\alpha$ is a volume form. Show that there exists a unique vector field $X$ on $M$ such that $\iota_X \alpha = 1$ and $\iota_X d\alpha = 0$. Furthermore, show that $\alpha \wedge d\alpha$ is invariant under $\varphi_t^X$, the flow generated by $X$.

**Problem 5.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse $C^\infty$ foliations on a connected manifold $M$ (i.e., foliations such that at every point $p \in M$, $\mathcal{F}_1(p) \cap \mathcal{F}_2(p)$, where $\mathcal{F}_i(p)$ is the leaf of $\mathcal{F}_i$ through $p$). Show that there exists a unique $C^\infty$ foliation $\mathcal{F}$ such that $\mathcal{F}(p) = \mathcal{F}_1(p) \cap \mathcal{F}_2(p)$.

**Problem 6.** Let $T^k = \mathbb{R}^k / \mathbb{Z}^k$, $f : T^k \to T^k$ be a $C^\infty$ map, and $U \subset T^k$ be an open set such that $f^{-1}(x)$ has exactly $m$ elements for every $x \in U$ and $\det(Df(x)) \det(Df(y)) \geq 0$ for all $x, y \in T^k$. Show that $\int_{T^k} f^* \omega = \pm m$, where $\omega = dx_1 \wedge \cdots \wedge dx_k$ is the standard volume form.