Problem 1 (20 points). Let \( A, B \subseteq \mathbb{R} \) be nonempty subsets, and assume that if \( x \in A \) or \( x \in B \) then \( x > 0 \). Show that if \( A/B = \{ x/y : x \in A \text{ and } y \in B \} \), then:
\[
\sup(A/B) = \frac{\sup A}{\inf B}
\]
whenever \( \inf B > 0 \).

Solution. Denote \( z = \frac{\sup A}{\inf B} \). We will show that \( z \) is an upper bound of \( A/B \), and that if \( y < z \), then \( y \) is not an upper bound of \( A/B \).

First, we show that \( z \) is an upper bound. If \( x \in A/B \), then there exists \( a \in A \) and \( b \in B \) such that \( x = a/b \). Since \( \sup A \) is an upper bound of \( A \), \( a \leq \sup A \). Similarly, \( b \geq \inf B \), and hence \( 1/b \leq 1/\inf B \) (since all elements of \( B \) are positive). Multiplying these inequalities, we see that
\[
a \leq \frac{\sup A \inf B}{z}.
\]
Hence \( z \) is an upper bound.

Now suppose that \( y < z \). If \( y \leq 0 \), \( y \) cannot be an upper bound since \( A \) and \( B \) consist of positive numbers. So without loss of generality, \( y > 0 \). Let \( \varepsilon = \frac{\sup A - y \inf B}{1 + y} \). Then \( \varepsilon > 0 \) since \( y < z \).

Since \( \sup A \) is the least upper bound of \( A \), there exists \( a \in A \) such that \( a > \sup A - \varepsilon \). Similarly, there exists \( b \) such that \( b < \inf B + \varepsilon \). Hence
\[
\frac{a}{b} > \frac{\sup A - \varepsilon}{\inf B + \varepsilon} = \frac{(1 + y) \sup A - (\sup A - y \inf B)}{(1 + y) \inf B + (\sup A - y \inf B)} = \frac{y(\sup A + \inf B)}{\inf B + \sup A} = y.
\]
Thus, \( y \) cannot be an upper bound of \( A/B \) and \( z = \sup(A/B) \). \( \square \)

Problem 2 (30 points, 10 each). Determine whether the sequence converges. If it converges, find its limit and prove that the sequence converges to that limit. If it diverges, prove that it diverges.

(a) \( \left\{ \frac{2n + 3}{8n + 7} \right\} \)
(b) \( \left\{ \frac{n^2 - 100n}{7n + 3} \right\} \)
(c) \( \{ \sin(\pi \cdot n) \} \)

Solutions.

(a) We claim this sequence converges to \( 1/4 \). This follows from the limit arithmetic theorem (aka, the main limit theorem):
\[
\lim_{n \to \infty} \frac{2n + 3}{8n + 7} = \lim_{n \to \infty} \frac{2 + 3/n}{8 + 7/n} = \frac{2}{8} \lim_{n \to \infty} \frac{1}{n} = 2/8 = 1/4.
\]
(b) We claim that the sequence diverges to $\infty$. Indeed, fix $M > 0$, and let $N = 7M$. Then if $n \geq N$,

$$\frac{n^2 - 100n}{7n + 3} \geq \frac{n^2}{7n} \geq \frac{(7M)^2}{7 \cdot (7M)} = M.$$ 

Hence the sequence diverges to $\infty$.

(c) Since $\sin(\pi \cdot n) \equiv 0$, the sequence is constant, and hence converges (to 0).

Problem 3 (Book 2.2.11, 20 points). Let $(a_n)$ and $(b_n)$ be sequences, and assume that $b_n \to 0$ and $|a_n| \leq b_n$ for every $n \in \mathbb{N}$. Prove that $a_n \to 0$.

Solution. Let $\varepsilon > 0$. Since $b_n \to 0$, there exists $N$ such that if $n \geq N$, $|b_n - 0| = |b_n| < \varepsilon$. Then if $n \geq N$,

$$|a_n - 0| = |a_n| \leq b_n \leq |b_n| < \varepsilon$$

Hence $a_n \to 0$.

Problem 4 (10 points). Show that if $I = [a, b]$ is a nonempty interval, and $x, y \in I$, then $|x - y| \leq b - a$.

Solution. If $x, y \in I$, then we have that $a \leq x, y \leq b$. Thus, we also know that $-x, -y \leq -a$. Adding these inequalities on the right in the nontrivial ways, we get that both $x - y \leq b - a$ and $y - x \leq b - a$. Hence $|x - y| \leq b - a$.

Problem 5 (20 points). Show that if $a_n \to L$ and $|b_n - a_n| \to 0$, then $b_n \to L$.

Solution. Let $\varepsilon > 0$. Since $a_n \to L$, there exists $N_1$ such that if $n \geq N_1$, $|a_n - L| < \varepsilon/2$. Similarly, there exists $N_2$ such that if $n \geq N_2$, $|b_n - a_n| < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$|b_n - L| = |(b_n - a_n) + (a_n - L)| \leq |b_n - a_n| + |a_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Hence $b_n \to L$. 

□