Problem 1. Prove the divergence criteria of the comparison test. That is, show that if \( \{a_n\} \) and \( \{b_n\} \) are sequences such that \( a_n \geq b_n \geq 0 \) for every \( n \), and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

Use the following scheme: Let \( t_m = \sum_{n=1}^{m} b_n \) be the sequence of partial sums for \( b_n \). Show that if \( \sum_{n=1}^{\infty} b_n \) diverges, then \( t_m \) must diverge to infinity. Then show that the sequence \( t_m \) diverges to \( \infty \), then so does \( s_m = \sum_{n=1}^{m} a_n \).

Problem 2. Determine whether the series converges or diverges. Prove that your answer is correct using the following tools only: the term test, the comparison test, the ratio test, and the integral test.

(a) \( \sum_{n=1}^{\infty} \cos(n) \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \)

(c) \( \sum_{n=1}^{\infty} \frac{n^8}{n!} \)

(d) \( \sum_{n=1}^{\infty} \frac{n^3 + 3^n}{5^n} \)

Problem 3. Let \( f_n : [a, b] \rightarrow \mathbb{R} \) be a sequence of positive functions such that \( \sum_{n=1}^{\infty} f_n(0) \) converges, and each \( f_n \) is \( L_n \)-Lipschitz. Show that if \( L = \sum_{n=1}^{\infty} L_n \) converges, then \( \sum_{n=1}^{\infty} f_n \) converges uniformly to an \( L \)-Lipschitz function.

Problem 4. Let \( f(x) = xe^x \), Find a number \( N \) such that the Taylor approximation of order \( N \) is within \( .1 \) of \( f(x) \) on the interval \([0, 1]\). \( \text{[Hint: First, find and prove a formula for } f^{(k)}(x) \text{ by induction, then bound } f^{(k)}(x) \text{ on } [0, 1] \} \text{ using this formula by a number depending on } k \)