Project Suggestions - pre-REU2024

Project 1 (Counting periodic orbits). This project aims to find both precise formulas and asymptotic estimates for the number of periodic orbits for piecewise expanding Markov interval maps. *Some linear algebra is required: matrix multiplication, eigenvalues, traces*

1. Given a graph Γ, define A_Γ to be the associated *adjacency matrix*. The entries of an adjacency matrix are always either 0 or 1. The $(i,j)^{\text{th}}$ entry is 1 if there is an edge connected i and j, and 0 if there is no such edge. For some piecewise expanding Markov interval maps, find the associated graph and adjacency matrix.

2. Show that the number of admissible words starting from i and ending at j of length ℓ is equal to the $(i,j)^{\text{th}}$ entry of $(A_\Gamma)^\ell$.

3. Show that (with the exception of finitely many words) there is a one-to-one correspondence between admissible words starting and ending with i of length ℓ, and points in I_i for which ℓ is a period.

4. Show that the number of periodic orbits of period ℓ for a piecewise expanding Markov interval map is the trace of $(A_\Gamma)^\ell$

5. Show that if A_Γ has a real eigenvalue $\lambda > 0$ which is larger than all others in absolute value (or modulus when complex), then

$$\lim_{\ell\to\infty} \frac{\mathrm{Tr}((A_\Gamma)^\ell)}{\lambda^\ell} = 1.$$

   We call such a λ the *exponent* for the growth rate.

6.* Look up and investigate the Perron-Frobenius theorem, and try to apply it here.

Project 2 (Constant slope models). This project aims to find "ideal" models for piecewise expanding Markov interval maps, so that each branch of the model has constant slope. *Some linear algebra is required: eigenvectors & eigenvalues*

1. Given a graph Γ, define A_Γ to be the associated *adjacency matrix*. The entries of an adjacency matrix are always either 0 or 1. The $(i,j)^{\text{th}}$ entry is 1 if there is an edge connected i and j, and 0 if there is no such edge. For some piecewise expanding Markov interval maps, find the associated graph and adjacency matrix.

2. If f is a piecewise expanding Markov interval map, let $v_i = \mathrm{length}(I_i)$. Show that v is an eigenvector of A_Γ.

3. Show that if f has constant slope, the slope must be an eigenvalue of A_Γ.

4. Show that if f has constant slope, then that slope must be a root of a polynomial with integer coefficients.

5. Consider a number of the form $\lambda = a + \sqrt{b}$, where a and b are both integers. Can you find a piecewise expanding Markov interval map with constant slope λ?

6.* Look up the Perron-Frobenius theorem, and try to apply it here. In particular, try to answer the question: when is a piecewise expanding Markov interval map conjugated to one with constant slope?

Project 3 (Graphing conjugacies)**.** This project aims to graph conjugacies for expanding circle maps and/or piecewise expanding Markov interval maps. *Familiarity with software capable of solving equations and creating piecewise graphs is required*

1. Recall the proof of the conjugacy between an expanding circle map of degree k and E_k, specifically that the conjugacy was increasing, and the image of a coding interval for f was the corresponding coding interval for E_k.

2. Find a way to express the endpoints of the coding intervals for f in terms of the one-sided inverses of f.

3. Show that you can find points on the graph of h, each of whose horizontal coordinate is the endpoint of a coding interval for f and the vertical coordinate is the endpoint of a coding interval for f.

4. Using the points which you know lie on the graph of h, write a computer program to draw an approximation of the graph of h, given a map f.

5. Try your program on the following families of examples for $\varepsilon \geq 0$:

$$f_\varepsilon(x) = \begin{cases} (2+\varepsilon)x, & 0 \leq x < 1/(2+\varepsilon) \\ \dfrac{(2+\varepsilon)x - 1}{1+\varepsilon}, & 1/(2+\varepsilon) \leq x < 1 \end{cases}$$

$$g_{k,\varepsilon}(x) = kx + \varepsilon \sin(2\pi x) - \lfloor kx + \varepsilon \sin(2\pi x) \rfloor$$

Project 4 (Obstructions to smooth conjugacy)**.** This project aims to understand when the conjugacy h between an expanding circle map and E_k is differentiable. This project will begin to work with a new key idea: a dynamical cocycle. Throughout, f denotes an expanding circle map.

1. We call a circle map *differentiable* if it is a continuous circle map which

   (a) is piecewise differentiable,

   (b) has that at each discontinuity, the left- and right-hand derivatives exist and are equal, and

   (c) the derivatives at 0 and 1 are equal.

   Justify this definition, and give examples indicating why each condition is important.

2. Show that the composition of differentiable circle maps is differentiable. You should need to use and conclude all 3 conditions.

   Let f be an expanding circle map. We say that f *is smoothly conjugated to* E_k if there exists a differentiable circle map $h : [0,1) \to [0,1)$ with differentiable inverse such that $h \circ f = E_k \circ h$.

3. Show that if f is smoothly conjugated to E_k, and p is a periodic point of f with ℓ as a period, then $(f^\ell)'(p) = k^\ell$.

4. Show that if there exists a continuous map $a : [0,1) \to \mathbb{R}$ such that $a(x) > 0$ for all $x \in [0,1)$ and $f'(x) = ca(x)/a(f(x))$ for a constant c, then $c = \deg(f)$. [*Hint*: Clear the denominators, then integrate both sides. Be careful with your bounds!]

5. Show that under the condition of the previous problem, f is smoothly conjugated to E_k for $k = \deg(f)$. [*Hint*: Would $h'(x)$ have to satisfy a similar equation? What reverses differentiation?]

6. Show that if f is smoothly conjugated to E_k, then the sequence $(f^n)'(x)/k^n$ is bounded away from 0 and ∞.

Project 5 (Orbit closures)**.** We showed that expanding circle maps have many periodic orbits. In this project, you will investigate what happens to other orbits. In particular, you will describe when a point's orbit is *dense* meaning the orbit "visits everywhere," and try to identify some orbits which are neither dense nor periodic.

1. Let $f : [0,1) \to [0,1)$ be a dynamical system. The *ω-limit set* of a point x is the set of all points $y \in [0,1)$ for which there exists an increasing sequence of natural numbers n_k such that

$$\lim_{k\to\infty} f^{n_k}(x) = y.$$

   The orbit closure of a point x is the union of the orbit of x with its ω-limit set. Try to understand and justify these definitions.

2. We say that an orbit is *dense* if its ω-limit set is equal to $[0,1)$. Show that if f and g are dynamical systems on $[0,1)$ which are conjugated by a continuous circle map, then f has a dense orbit if and only if g has a dense orbit.

3. Show that E_k has a dense orbit for all $k \geq 2$.

4. Conclude that any expanding circle map has a dense orbit.

5. Find a point whose orbit is neither periodic nor dense.

6. Look up the (middle-thirds) Cantor set, and obtain (almost) this set as an orbit closure for E_3.

Project 6 (The Gauss map and continued fraction digit expansions)**.** This project aims to understand the "infinite degree" map $G(x) = 1/x - \lfloor 1/x \rfloor$, and what sorts of digit expansions is associated to it. Numbers with periodic expansions will have a special (but different!) property.

1. Graph $G : (0,1) \to [0,1)$. Show that all discontinuities of G are compatible with the definition of an expanding circle map, but cannot be defined at 0 in a way that makes it a continuous circle map.

2. Find the continuity domains and one-sided inverses of G (there will be infinitely many!).

3. Show that if $x \in (0,1)$ is rational, $G^k(x) = 0$ for some k, and if $x \in (0,1)$ is irrational G can be iterated indefinitely.

4. Show that $x \in (0,1)$ is pre-periodic for G if and only if x can be expressed as $a + \sqrt{b}$ for some rational numbers a and b such that b is not a perfect square.

5. Show that every irrational $x \in (0,1)$ has a sequence of natural numbers a_i such that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

   where this expression is interpreted as

$$x = \lim_{n\to\infty} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \cfrac{+1}{a_n}}}}$$

Project 7 (Ordering Coding Intervals)**.** This project aims to describe the ordering of coding intervals when a piecewise expanding Markov interval is not increasing. You can then drop the increasing condition in the classification theorem. You will produce a new ordering (other than the lexicographical ordering) on admissible finite words associated to a piecewise expanding Markov interval map.

1. Start by considering the "tent map"

$$T(x) = \begin{cases} 2x, & [0,1/2) \\ 2-2x, & [1/2,1] \end{cases}$$

   Show that T is a continuous map of $[0,1]$ (not just a continuous circle map!), and that T has one-sided inverse on (now overlapping) coding intervals.

2. Show that the endpoints of the coding intervals for T are the same as the endpoints for E_2, but they are listed in a different order.

3. Find a way to describe the order in which the coding intervals appear for T.

4. Let f be a piecewise expanding Markov interval map on $[0,1]$, and label each coding interval with a + or - depending on whether f is increasing or decreasing on it. We call this the *sign labeling.* Find a way to describe the order of the coding intervals of f.

5. Show that any two piecewise expanding Markov interval maps with the same sign labeling are conjugated by a continuous map of $[0,1]$.

Project 8 (Toral automorphisms)**.** This project extends parts of what we did for expanding circle maps to a higher-dimensional analog: toral automorphisms. *Some linear algebra is required: eigenvalues*

1. The *flat torus* is obtained from taking a square and "gluing" opposite sides. Explain why this should be thought of as "$0=1$" in both the horizontal and vertical coordinates of $[0,1)^2 = \{(x,y) : x,y \in [0,1)\}$. If $(x,y) \in \mathbb{R}^2$, define $\lfloor\lfloor(x,y)\rfloor\rfloor = (\lfloor x\rfloor, \lfloor y\rfloor)$. Show that $\lfloor\lfloor(x,y)\rfloor\rfloor$ is always in $[0,1)^2$.

2. A map $f : [0,1)^2 \to [0,1)^2$ is a called a continuous map of the flat torus if there exists a continuous map $F : [0,1]^2 \to \mathbb{R}^2$ such that

$$f(x,y) = F(x,y) - \lfloor\lfloor F(x,y)\rfloor\rfloor$$

   and $F(1,y) - F(0,y) \in \mathbb{Z}$ for all $y \in [0,1)$ and $F(x,1) - F(x,0) \in \mathbb{Z}$ for all $x \in [0,1)$. Compare this with what we know about continuous circle maps, and explain why it is a reasonable definition.

3. Show that if g_1 and g_2 are continuous circle maps, then $f(x,y) := (g_1(x), g_2(x))$ is a continuous map of the flat torus.

4. Suppose that A is a 2×2 matrix, and f be the map of the flat torus constructed from A as in part 3. Show that f is a continuous circle map if and only if every entry of A is an integer. f is called the *system induced by A.*

5. Show that if A is a 2×2 matrix with integer entries and two real eigenvalues distinct from 0, 1 and -1, then the set of periodic orbits for the induced system f is exactly the set of points with rational entries.

6. Show that the system induced by an integer matrix A is invertible if and only if $\det(A) = \pm 1$.