Problem Set 4 - pre-REU 2024

1. Prove the main classification theorem for Markov dynamics:

   Theorem. Let f_1 and f_2 be piecewise increasing, expanding, Markov interval maps on $[0, 1)$. If f_1 and f_2 have the same associated graph (with the same labeling), then f_1 and f_2 are conjugated by an increasing bijection $h : [0, 1) \to [0, 1)$.

2. Give an example of an expanding, piecewise increasing, Markov system for which there is more than one forbidden terminating repeated digit.

3. Give an example of a piecewise increasing, expanding, Markov system for which every periodic orbit has even period.

4. Show that any piecewise increasing Markov system has a periodic orbit.

5. A piecewise differentiable system $f : [0, 1) \to [0, 1)$ is called *eventually expanding* if there exists some $\lambda \in (1, \infty)$ and $n \in \mathbb{N}$ such that $(f^n)'(x) \geq \lambda$ for all $x \in [0, 1)$. Show that the main classification theorem still holds for eventually expanding systems.

6. Give an example of a piecewise increasing system which is Markov and eventually expanding, but not expanding.

7. Give an example of a piecewise increasing system which is Markov, but not eventually expanding.

8. Show that any continuous circle map of degree k has the Markov property relative to its coding intervals.

9. A *self-conjugacy* of a dynamical system $f : X \to X$ is a map $h : X \to X$ such that $h \circ f = f \circ h$. Find all continuous self-conjugacies of E_k. (* If you're feeling bold, try to prove that you answer is correct)

10. * Consider a piecewise increasing, expanding interval map $f : [0, 1) \to [0, 1)$ with discontinuities $d_1, \ldots, d_{k-1}$, let $d_0 = 0$, and $d_k = 1$. An *post-discontinuity* is any endpoint of $f^n([d_i, d_{i+1}))$ for some $i = 1, \ldots, k-1$, $n \geq 1$. Show that f is Markov relative to its continuity domains if and only if every post-discontinuity is a discontinuity.

11. * Show that if there are only finitely many post-discontinuities, listed $\{0 = b_0, b_1, \ldots, b_\ell = 1\}$, then f has the Markov property relative to the intervals $I_j = [b_j, b_{j+1})$.

12. * Give an example of a piecewise increasing, expanding interval map which is not Markov relative to its continuity domains, but it Markov relative to another collection of intervals.

13. * Fix a <u>real</u> number $\beta > 1$. Consider the piecewise increasing, expanding interval transformation

$$f(x) = \beta x - \lfloor \beta x \rfloor$$

    For which values of β is this transformation Markov relative to some collection of intervals?