Problem Set 3 - pre-REU 2024

Definition 1. A map $f : [0,1) \to [0,1)$ is a *continuous circle map* if

(i) either $\lim_{x\to 1^-} f(x) = f(0)$ or $\lim_{x\to 1^-} f(x) = 1$ and $f(0) = 0$, and

(ii) f has finitely many discontinuities $a_1, \ldots, a_n$, and at each such discontinuity. both

$$\lim_{x\to a_i{}^+} f(x) \text{ and } \lim_{x\to a_i{}^-} f(x)$$

are equal to either 1 or 0, and $f(a_i) = 0$.

1. Finish the proof of the conjugacy for two increasing contractions by building the conjugacy for points to the left of the fixed points.
2. Show that if $\mathrm{Pre}(f) = \{x \in [0,1] : x \text{ is preperiodic for } f\}$, then $\mathrm{Pre}(E_k) = [0,1) \cap \mathbb{Q}$ whenever $k \geq 2$.
3. Fix k and q. How many points does E_k have such that q is a period?
4. Prove that $E_k(E_\ell(x)) = E_{k\ell}(x)$ for all $x \in [0,1)$.
5. Show that for every $\alpha \in [0,1)$, the function $h(x) = x + \alpha - \lfloor x + \alpha \rfloor$ is invertible, and both h and its inverse are continuous circle transformations.
6. Let $F : [0,1] \to \mathbb{R}$ is a continuous map. Show that if for every $y \in \mathbb{R}$, $\#\{x \in [0,1) : F(x) = y\}$ is a finite set and $F(1) - F(0) \in \mathbb{Z}$, then $f(x) = F(x) - \lfloor F(x) \rfloor$ is a continuous circle map. Furthermore, show that if F is increasing, then f is piecewise increasing and $\deg(f) = F(1) - F(0)$.
7. Show that if f and g are continuous circle transformations and for every $y \in [0,1)$, $\#\{x \in [0,1) : f(x) = y\}$ is a finite set, then $h(x) := g(f(x))$ is a continuous circle transformations. What are the possible discontinuity points of h? [*Hint*: Try composing the functions E_k with the rotations of problem 5 to get some intuition.]
8. Show that there does not exist a continuous circle map f such that f is continuously differentiable and $0 < f'(x) < 1$ for all $x \in [0,1)$.
9. Show that if f is an expanding circle map, then $\deg(f) \geq 2$.
10. Find a formula for $(f^k)'(x)$ in terms of the derivative of f itself, and prove your formula. [*Hint*: Use induction and the chain rule]
11. Show that if f is an expanding circle map, then there exists a rotation function h as in problem 5 such that $g(x) := h^{-1}(f(h(x))$ has a fixed point at 0.
12. Define

$$f(x) = \begin{cases} 3x/2, & 0 \leq x < 2/3 \\ 3x - 2, & 2/3 \leq x < 1 \end{cases}$$

Find the degree of f, the one-sided inverses of f, and the 2- and 3-step coding intervals for f.