Problem Set 2 - pre-REU 2024

1. Fix $c \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that $|\lambda| < 1$. Show that the dynamical system $T(x) = \lambda x + c$ has a unique fixed point p, and for every $x \in \mathbb{R}$, $\lim_{n\to\infty} T^n(x) = p$.

2. Show that if $f : X \to X$ is a dynamical system on a finite set X, then the following are equivalent:

   (a) f is injective.

   (b) f is surjective.

   (c) Every point of X is f-periodic.

   (d) There exists a dynamical system $g : X \to X$ such that $g(f(x)) = x = f(g(x))$ for all $x \in X$.

3. Show that if X is not finite, then we still have that

   - (d) implies (a)
   - (d) implies (b)
   - (c) implies all other properties

4. Show that the implications of the previous problem are the only ones to pass to infinite systems. That is, for every pair of distinct properties (♠) and (♣) (chosen from (a)-(d), omitting the 5 exceptions), find a dynamical system on an infinite set which has property (♠), but not property (♣).

   [*Note*: This is 7 problems in 1! Feel free to do a few and move on to finish later if you have time. Some counterexamples may work for more than one pair.]

5. Does the nested interval property hold for intersections of nested half-open intervals $I_n = [a_n, b_n)$? If not, is there a condition you can add to the assumptions to make it true? (Don't try to prove your answer)

6. Let I_n is a sequence of nested closed intervals. If there exists only one point p such that $p \in I_n$ for every n, show that $\ell(I_n) \to 0$.

7. Let $[a, b]$ be a closed interval, and $f : [a, b] \to [a, b]$ be a continuous dynamical system. Does a periodic point always exist? If not, find a counterexample. If so, prove that it does!

8. Let $f : X \to X$ be a dynamical system, and $A \subset X$ be a subset such that $f(A) \subset A$. Show that for every $k \in \mathbb{N}_0$, $f^{k+1}(A) \subset f^k(A)$. [*Hint*: First show that whenever $A \subset B$, $f(A) \subset f(B)$, then run a proof by induction]

9. Let $p \in \mathbb{R}$ be a fixed point of a dynamical system $f : \mathbb{R} \to \mathbb{R}$. p is called *attracting* if there exists an interval $[a, b]$ such that $p \in (a, b)$, and for all $x \in [a, b]$, $\lim_{n\to\infty} f^n(x) = p$. Show that if f is continuously differentiable and $|f'(p)| < 1$, then p is attracting.

10. Is it possible for a continuous dynamical system $f : [0, 1] \to [0, 1]$ to have a point of period 2? Can it have a point of period q for all $q \in \mathbb{N}$?

11. Does the contraction mapping principle hold if $|T'(x)| < 1$ for all $x \in \mathbb{R}$? (Start by comparing this assumption with the definition of a contraction)