Problem Set 1 - pre-REU 2024

1. Find the fixed points of $T(x) = x^2$ on $\mathbb{R}$.

2. Fix $p, q \in \mathbb{N}$. If $A = \{k \in \mathbb{N}_0 : kq \leq p\}$, n is the maximal element of A and $r = p - nq$, show that $0 \leq r < q$.

3. Show that if x is fixed by T, then $T^n(x) = x$ for all $n \in \mathbb{N}$.

4. Show that if x is periodic for T, then there exists $q \in \mathbb{N}$ such that $\{n \in N : T^n(x) = x\} = \{kq : k \in \mathbb{N}\}$.

5. Show that if x is T-periodic, then $T^k(x)$ is T-periodic for all $k \in \mathbb{N}_0$.

6. Show that if $T^k(x)$ is periodic, then x is not necessarily periodic.

7. Let x be pre-periodic. Show that for every $r \in \mathbb{N}_0$ and $A_r := \{n \in \mathbb{N} : T^n(x) = T^r(x)\}$ is either $\{r\}$ or there exists $r' \in \mathbb{N}$ such that

$$A_r = \{kq + r' : k \in \mathbb{N}_0\}$$

8. Prove the exponent rule $T^{m+n}(x) = T^m(T^n(x))$ [*Hint*: Find a variable to induct on]

9. Let $T(x)$ by given by a polynomial of degree d. What is the degree of the polynomial expression for $T^n(x)$?

10. Fix q and n. Consider the space $X = \{0, \ldots, q-1\}$ and the dynamical system

$$T(x) = \text{the remainder of } nx \text{ after division by } q$$

Draw some pictures describing these dynamical systems for a few different values of n and q. Do you have any conjectures about which points will be periodic for T?