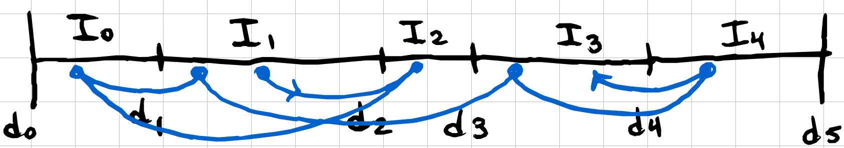
Day 10 (26 July 2024)

Last time: $f:[0,1)\to[0,1)$ expanding, $f(0)=0$
↳ coding intervals of length 1 are the continuity domains:



- The interval $[0,1)$ is a disjoint union of the I_k's with no overlaps: ← "union with no overlap"
$$[0,1) = I_0 \sqcup I_1 \sqcup I_2 \sqcup I_3$$
- Each x has an itinerary: Define a_n to be the number such that $f^n(x)\in I_{a_n}$.
- $w(x)$ = word associated to x
$= 1201343\ldots$

<u>Lemma</u> Let $w=a_1\ldots a_\ell$ denote a fixed word. Then
$$\{x\in[0,1) : f^{n-1}(x)\in I_{a_n}\}$$
$$= g_{a_1}(g_{a_2}(\ldots(g_{a_n}([0,1)))\ldots)) =: I_w.$$

Observations:
- length$(I_w)\le \lambda^{-\ell}$ where ℓ = length(w).
- if w is a prefix of v (that is, v starts with w) then $I_v \subseteq I_w$. This allows us to construct a nested interval property!
- if w is an infinite word and w_n is its truncation up to length n: $w_n = a_1a_2\ldots a_n$, then the righthand endpoint moves unless the added digit is $K-1$. proof on next page ⊛

Proof of ⊛: Let $I_{a_1 \ldots a_{n+1}} \in [a,b)$

$$= g_{a_1}(g_{a_2}(\ldots(g_{a_{n+1}}([0,1)))\ldots)).$$

Then $b = g_{a_1}(g_{a_2}(\cdots(g_{a_{n+1}}(1))\ldots))$. If the righthand endpoint doesn't move, then

$$g_{a_1}(g_{a_2}(\cdots(g_{a_{n+1}}(1))\ldots))$$
$$= g_{a_1}(g_{a_2}(\ldots(g_{a_n}(1))\ldots)).$$

Applying f to both sides n times yields

$$g_{a_{n+1}}(1) = 1.$$

Hence, since $g_{a_{n+1}}([0,1)) = I_{a_{n+1}}$, 1 is in the (limits of the) image $\Rightarrow I_{a_{n+1}}$ is the last coding interval $\Rightarrow a_{n+1} = k-1$.

END OF RECAP.

Now we finish the theorem of the week. We start by defining the conjugacy: If $x \in [0,1)$, let

$$w(x) = a_1 a_2 \ldots a_n \ldots$$

be the infinite code of x. Define

$$h(x) := \sum_{i=1}^{\infty} \frac{a_i}{k^i}$$

where $k = \deg(f)$. We want to check that

- h satisfies the conjugacy equation with E_k
- h is bijective
- h is increasing

<u>Conjugacy</u>: $h(f(x)) = E_k(h(x))$

← Claim:

$$E_k(h(x)) = E_k\left(\textstyle\sum_{i=1}^{\infty} (a_i/k^i)\right)$$
$$= k\,\Sigma \text{ '' '' } - \lfloor k\,\Sigma \text{ '' ''} \rfloor$$
$$= \left(\sum_{i=1}^{\infty} \frac{a_i}{k^{i-1}}\right) - \lfloor \text{ '' '' } \rfloor$$
$$= \left(\sum_{i=2}^{\infty} \frac{a_i}{k^{i-1}} + a_1\right) - a_1$$
$$= \sum_{i=1}^{\infty} \frac{a_{i+1}}{k^i}$$

$$= \textstyle\sum_{i=1}^{\infty} (a_i/k^{i-1})$$
$$= a_1 + (a_2/k) + (a_3/k^2) + \ldots$$
$$< a_1 + (k-1)\textstyle\sum_{i=1}^{\infty} (1/k^i)$$
$$= a_1 + 1.$$

Strict inequality due to ⊛

If $a_1 \dots a_n \dots$ is the code of x, the code of $f(x)$ is $a_2 \dots a_n \dots$. Indeed,

$$f^{n-1}(x) \in I_{a_n} \Rightarrow f^n(x) \in I_{a_{n+1}},$$

so $f^{n-1}(f(x)) \in I_{a_{n+1}}$. ("f shifts by 1")

Now what is $h(f(x))$? Well,

$$h(f(x)) = \sum_{i=1}^{\infty} \frac{a_{i+1}}{k^i}$$

because we are simply shifting by 1. Since we've shown that $h(f(x)) = E_k(h(x))$, h satisfies the conjugacy equation.

<u>Bijectivity</u>: Claim – h is bijective

- surjectivity: Let $y \in [0,1)$ and let a_k be the itinerary of y under E_k. By ⊛, a_k does not end in all $(k-1)$'s. Let w_n denote the truncation $w_n = a_1 a_2 \dots a_n$, and I_{w_n} denote the corresp. interval for f. Since the righthand endpoint moves infinitely often, $\exists!$ (there exists unique) x such that $x \in I_{w_n}$ $\forall n$. Then

$$h(x) = \sum_{i=1}^{\infty} \frac{a_i}{k^i} = y.$$

↳ uniqueness not actually necessary here, though we achieve it

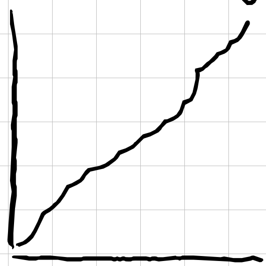
- injectivity: Suppose $h(x) = h(y)$. We will show that $x = y$. Since $h(x) = h(y)$,

$$\sum_{i=1}^{\infty} \frac{a_i}{k^i} = \sum_{i=1}^{\infty} \frac{b_i}{k^i}$$

where $a_1 \dots a_n \dots$ and $b_1 \dots b_n \dots$ are the codes of x and y, respectively.
<u>Exercise</u>: Base k digit expansions are unique so long as they don't terminate in all $(k-1)$'s.
Hence, $a_i = b_i$ $\forall i$ and so $x, y \in I_{w_n}$ $\forall n$ and so $|x-y| \leq \lambda^{-i}$ $\forall i \in \mathbb{N}$ and so $|x-y| = 0$ and so $x = y$.

<u>Increasing</u>: Claim – h is increasing

Sketch $\rightsquigarrow$ h looks something like



We want to show
- h increasing
- $h(0) = 0$
- $\lim_{x \to 1^-} h(x) = 1$

<u>Exercise</u>: formalize this!

increasing: Take $x \neq y$. Then $\exists n$ such that $a_n \neq b_n$ while $a_m = b_m$ $\forall m < n$, where $a_1 \ldots a_n \ldots$ and $b_1 \ldots b_n \ldots$ are the codings of x and y. Since the intervals are ordered like #s, if $x < y$ then $a_n < b_n$ and so $h(x) < h(y)$

$h(0) = 0$: This is just a computation. Since $f(0) = 0$, the code of 0 is $0000\ldots$

$\lim_{x \to 1^-} h(x) = 1$: The coding intervals whose endpoints are 1 have only $(k-1)$'s by $\circledast$. Hence, if $x \to 1^-$ then its coding begins with longer strings of $(k-1)$'s. By the definition of h, $h(x) \to 1$.