Recall. Last time, we proved a lemma to set up for our next big theorem. We'll restate them both here.

Lemma. *If f is an expanding circle map of degree k, then f has a fixed point (in fact, $k-1$ fixed points).*

Theorem. *If f is an expanding circle map of degree k, then f is conjugated to E_k by an invertible continuous circle map whose inverse is a continuous circle map.*

Remark. A corollary is like a theorem or lemma, but it has a proof that's relatively short using a previous theorem or lemma. A sketch of a proof is the general outline of how to write a proof without any of the details filled in. This can be useful when giving the proof as an exercise or if we believe something is true and plan to get back to the full proof later.

Corollary. *If f is an expanding circle map, f is conjugated to an expanding circle map with a fixed point at zero.*

Sketch. Idea: "change the cut point to zero."

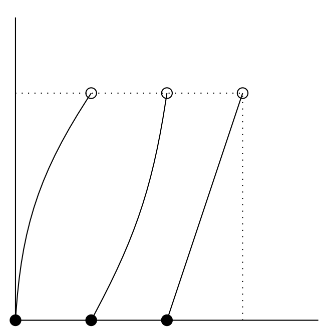
Let p be a fixed point for f, and $h(x) = x + p - \lfloor x + p \rfloor$. Then if $g(x) = h^{-1}(f(h(x)))$, g is expanding and $g(0) = 0$. □

★**Exercise.** *Check the details of this sketch.*

★**Exercise.** *Prove the following: If f is conjugated to g, and g is conjugated to ℓ, then f is conjugated to ℓ.*

Remark. The moral of the previous corollary and exercises is that when proving the main theorem, we can assume without loss of generality that $f(0) = 0$.

Remark. Now, let $d_1, \ldots, d_{k-1}$ denote the discontinuities of f, let $d_0 = 0$, and let $d_k = 1$. Consider the graph:



Notice that since $f(0) = 0$, the first discontinuity must jump from 1 to 0, so $\lim_{x \to d_1^-} f(x) = 1$. Similarly, on the last interval, f must tend to either zero or one, but since it is expanding (so increasing), it must limit to 1. Thus, $f : [d_i, d_{i+1}) \to [0, 1)$ is an increasing bijection for each i.

Let $g_i : [0,1) \to [d_i, d_{i+1})$ denote the inverse of the *restriction* of f to $[d_i, d_{i+1})$. We call g_i the one-sided inverses of f. Indeed,

$$\forall x \in [0,1) \mid f(g_i(x)) = x$$
$$\color{red}{\forall x \in [d_i, d_{i+1})} \mid g_i(f(x)) = x$$

In particular, the second equation doesn't hold for all inputs x.

Example. $E_{10}(x) = 10x - \lfloor 10x \rfloor$. Then $d_i = i/10$ for $i = 0, \ldots, 10$. For $x \in [d_i, d_{i+1})$, we have

$$E_{10}(x) = 10x - i$$

so we can find the inverse on this region by solving $x = 10y - i$. I.e.

$$g_i(x) = \frac{x}{10} + \frac{i}{10}$$

We can check the claimed properties now. For $x \in [0,1)$:

$$\begin{aligned}
f(g_i(x)) &= f\left(\frac{x}{10} + \frac{i}{10}\right) \\
&= 10\left(\frac{x}{10} + \frac{i}{10}\right) - \left\lfloor 10\left(\frac{x}{10} + \frac{i}{10}\right)\right\rfloor \\
&= x + i - \lfloor x + i \rfloor \\
&= x + i - i \\
&= i \\
\text{and} \\
g_i(f(x)) &= g_i(10x - \lfloor 10x \rfloor) \\
&= \frac{10x - \lfloor 10x \rfloor}{10} + \frac{i}{10} \\
&= x + \frac{i - \lfloor 10x \rfloor}{10}
\end{aligned}$$

which may not equal x – we could be off by an integer multiple of $1/10^{\text{th}}$.

In fact, we can understand this directly. If we write out the decimal expansion of x as:

$$x = 0.u_1 u_2 u_3 \ldots$$

then

$$E_{10}(x) = 0.u_2 u_3 u_4 \ldots$$

An inverse function should "undo" this and go backwards. I.e. it should shift the digits to the right and put the first digit back in. But there are many choices for the first digit. g_i specifically picks the digit i:

$$g_i(x) = 0.i u_1 u_2 u_3 \ldots$$

We'll sometimes call this an insertion operation since we've inserted the digit i.

Combinatorial words and Coding Intervals

We continue to work with an expanding circle map f of degree k with one-sided inverses $g_0, \ldots, g_{k-1}$.

Let $A_0 = 0, \ldots, k-1$ denote the "standard alphabet" on k letters. A finite word in A_0 is a list of letters $w = a_1 a_2 \ldots a_\ell$.

Remark. We should not think of this as the same as the word with infinitely many trailing zeros. That will be a distinct word (this is different from our intuition about decimals).

Definition. The coding interval of f corresponding to w is the set

$$I_w = g_{a_1}(g_{a_2}(\cdots(g_{a_\ell}([0,1)))\cdots))$$

Lemma. *For f and w as above,*

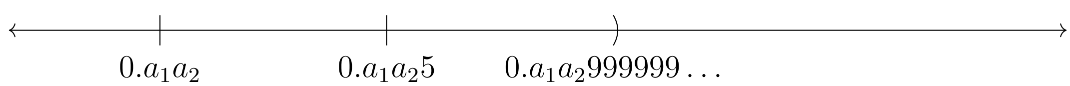
$$I_w = \{x \in [0,1) : f^{i-1}(x) \in [d_{a_i}, d_{a_i+1})\forall i = 1, \ldots, \ell\}$$

Furthermore, for a fixed ℓ, $[0,1)$ is the union of all coding intervals for words of length ℓ.

Example. Let's look again at E_{10}. Then,

$$\begin{aligned} g_{a_1}(g_{a_2}(x)) &= g_{a_1}(x/10 + a_2/10) \\ &= (x/10 + a_2/10)/10 + a1/10 \\ &= \frac{x}{100} + \frac{a_1}{10} + \frac{a_2}{100} \end{aligned}$$

So, $I_{a_1a_2}$ is the set of points whose first two digits are a_1a_2.



I.e. this is an interval of length $\frac{1}{100}$.

★**Exercise.** *Prove the previous lemma.*

Proof of main theorem. Recall our assumption is that f is expanding.

Suppose $x \in [0,1)$. Let w_n denote the (unique) word of length n such that $x \in I_{w_n}$.

CLAIM 1: w_{n+1} is the extension of w_n by one letter on the right. This follows from the lemma since $f^{i-1}(x)$ is prescribed by x for $i = 1, \ldots, n$.

CLAIM 2: $\mathrm{length}(I_w) \leq \lambda^{-n}$, where $\lambda \in (1, \infty)$ is such that $f'(x) \geq \lambda$ for all x.

Indeed, differentiating $f(g_i(x)) = x$ gives $f'(g_i(x))g_i'(x) = 1$, so

$$g_i'(x) = \frac{1}{f'(g_i(x))} \leq \frac{1}{\lambda}$$

Hence, for any x,

$$(g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})'(x) \leq \frac{1}{\lambda^n}$$

So,

$$\begin{aligned}
&\mathrm{length}(I_{w_n}) \\
&\quad = (g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})(1) - (g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})(0) && \text{[increasing functions]} \\
&\quad = \frac{(g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})(1) - (g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})(0)}{1-0} \\
&\quad = (g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_n})'(c) && \text{for some } c \text{ [MVT]} \\
&\quad \leq \lambda^{-n}
\end{aligned}$$

This proves claim 2. Note that as a result, given an infinite word $w = a_1 a_2 a_3 \ldots a_n \ldots$, there is at most one x in every I_{w_n}, where w_n is the finite truncation $w_n = a_1 \ldots a_n$.

CLAIM 3: If w is a word that does not terminate in a repeating $k-1$ (i.e. it is not the case that it ends with the digit $k-1$ repeatedly after some finite prefix), there exists some point x such that $x \in I_{w_n}$ for every n.

This is true by the nested interval property and the fact that the right endpoint of I_{w_n} equals the right endpoint $I_{w_{n+1}}$ if and only if $a_{n+1} = k-1$. Let's prove this "if and only if" subclaim.

Note:

$$I_{w_{n+1}} = g_{a_1}(g_{a_2}(\cdots g_{a_n}(g_{a_{n+1}}([0,1))) \cdots))$$

Comparing the right endpoints, we're hoping that

$$g_{a_1}(g_{a_2}(\cdots g_{a_n}(g_{a_{n+1}}(1)) \cdots)) = g_{a_1}(g_{a_2}(\cdots g_{a_n}(1) \cdots))$$

Applying f^n to both sides, we're hoping that

$$g_{a_{n+1}}(1) = 1$$

Since the image of $g_{a_{n+1}}$ is the $(a_{n+1})^{\text{th}}$ coding interval, this is only possible for the last interval, i.e. $a_{n+1} = k-1$. Conversely, if $a_{n+1} = k-1$, then $g_{a_{n+1}}(1) = 1$ since this is the chosen fixed point, and the argument works in reverse. This proves the subclaim and thus claim 3.

Therefore, for every $x \in [0,1)$, we have shown there exists a unique code $w = a_1 a_2 a_3 \ldots$. Define

$$h(x) = \sum_{n=1}^{\infty} \frac{a_n}{k^n}$$

Tomorrow, to finish the proof, we will show that h is a conjugacy and that h is a continuous circle map. $\square$