Day 7 (23 July 2024)

<u>Theorem</u> | If f is an expanding circle map of degree k, then $\exists$ a continuous circle map h such that h conjugates f and E_k.

$\hookrightarrow$ Theorem of the week!

<u>Lemma</u> | If f is an expanding circle map then f has a fixed point.

Pf of lemma: Define $h(x) = f(x) - x - \lfloor f(x) - x \rfloor$. Claim $\circledast$: h is an increasing continuous circle map & $\deg(h) = \deg(f) - 1 \geq 1$. Using the claim, $\exists$ at least one point p such that $h(p) = 0$

$$\Rightarrow \quad f(p) - p = \lfloor f(p) - p \rfloor.$$

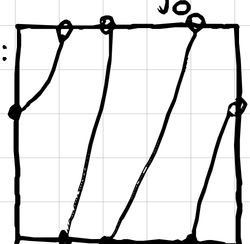
Since both $f(p), p \in [0,1)$, $f(p) - p \in (-1,1)$ and so its only integer solution is zero.

(Exercise: If f and g are increasing continuous circle maps, so is $h(x) = f(x) + g(x) - \lfloor f(x) + g(x) \rfloor$ and $\deg(h) = \deg(f) + \deg(g)$).

<u>Formula</u> | If f is a differentiable increasing circle map, then $\deg(f) = \int_0^1 f'(x)\,dx \neq f(1) - f(0)$

Example: Consider E_k. $E_k' = k$. Hence, $\int_0^1 k\,dx = k = \deg(E_k)$.

Pf of formula:



By Fundamental Thm of Calc, $\int_{a_i}^{a_{i+1}} f'(x)\,dx = 1$ on each middle domain.

End intervals: $\int_{a_1}^{a_2} f'(x)\,dx = 1 - f(0)$, $\int_{a_n}^{a_{n+1}} f'(x)\,dx = f(0)$

Pf of claim ⊛: Recall $h(x) = f(x) - x - \lfloor f(x) - x \rfloor$.

Since f is expanding, $\exists \lambda \in (1,\infty)$ such that $f'(x) \geq \lambda \ \forall x$. Hence,

$$h'(x) = f'(x) - 1 \geq \lambda - 1 > 0,$$

so h is increasing on each continuity domain. And so if h is a continuous circle map,

$$\deg(h) = \int_0^1 h'(x)\,dx$$

$$= \int_0^1 f'(x) - 1 \, dx = \deg(f) - 1.$$

It remains to show that indeed h is a continuous circle map: Possible discontinuities of h occur at (A) discontinuities of f and (B) discontinuities of the floor function.

TYPE A: Let a denote a discontinuity point for f. Then $\lim_{x \to a^{\pm}} f(x) = 0$ or 1 and $f'(a) = 0$.

Hence, $\lim_{x \to a^{\pm}} h(x) = \lim_{x \to a^{\pm}} (f(x) - x - \lfloor f(x) - x \rfloor)$

$$= (0 \text{ or } 1) - a - \lfloor (0 \text{ or } 1) - a \rfloor$$

Zero case: $= 0 - a - \lfloor 0 - a \rfloor = -a + 1$ $\Big\} =$

One case: $= 1 - a - \lfloor 1 - a \rfloor = 1 - a$

So the left- and right-hand limits are the same, at least when $a \in (0,1)$, which is exactly what we are concerned with since discontinuities of f are between zero and one.

TYPE B: These occur when $f(x) - x \in \mathbb{Z}$. We already showed that the only such integer is zero:

$$f(x) - x = 0.$$

Pick a point $b \in [0,1)$ such that $f(b) = b$.

Then $\lim_{x \to b^{\pm}} h(x) = \lim_{x \to b^{\pm}} f(x) - x - \lfloor f(x) - x \rfloor$

$= 0 - (0 \text{ or } -1)$

$= 0 \text{ or } 1,$

which is what we want.

Final step: We need to show that

EITHER $\lim_{x \to 1^-} h(x) = h(0)$ $\quad$ I

OR $\lim_{x \to 1^-} h(x) = 1$ and $h(0) = 0.$ $\quad$ II

$\ldots \lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} f(x) - x - \lfloor f(x) - x \rfloor$

If f satisfies I

$= f(0) - 1 + 1$

$= f(0)$

$= h(0)$

If f satisfies II

$= 1 - 1 + 1$

$= 1$ because f is expanding

(and $h(0) = 0$)