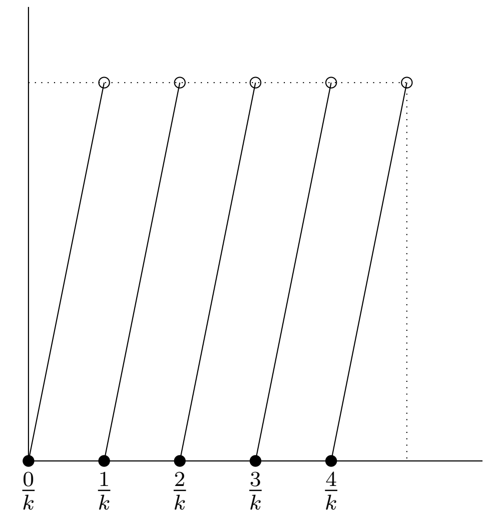
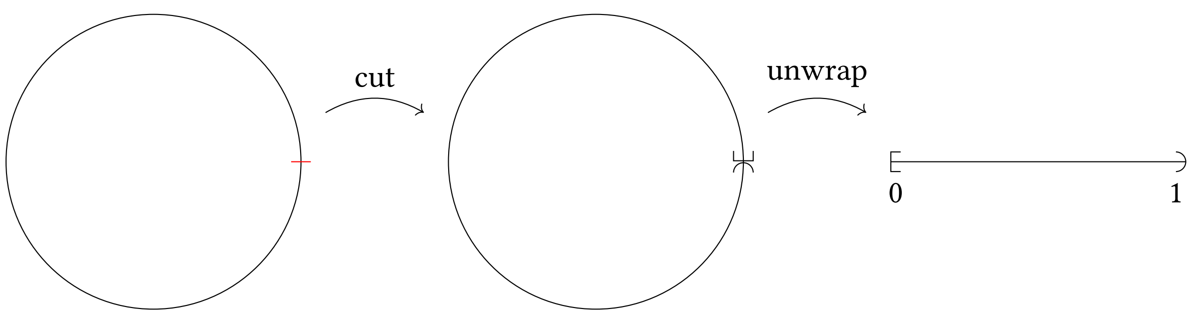
Recall. $E_k(x) = kx - \lfloor kx \rfloor$. Picture of E_5:



This has discontinuities at $\frac{n}{k}, 1 \leq n \leq k-1$. All discontinuities have the feature that

$$\lim_{x \to a^-} f(x) = 1$$
$$\lim_{x \to a^+} f(x) = 0$$

If "$1 = 0$," these discontinuities wouldn't happen! Idea: Introduce a new continuity criterion where "$1 = 0$." Let X be a circle of circumference 1.



Definition. $f : [0,1) \to [0,1)$ is a continuous circle map if:

- $\lim_{x \to 1^-} f(x) = f(0)$ OR $\lim_{x \to 1^-} f(x) = 1$ and $f(0) = 0$.
- There are finitely many discontinuities $a_1, \ldots, a_\ell$ of f, and either

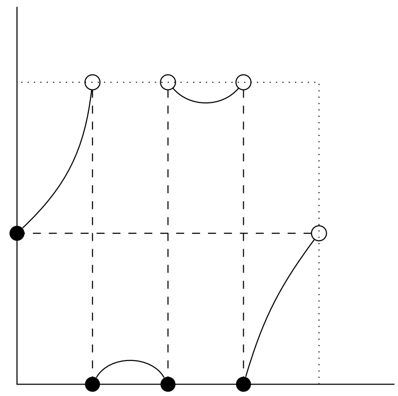
  $$\lim_{x \to a_i^-} f(x) = 1 \text{ and } \lim_{x \to a_i^+} f(x) = 0$$

  or

  $$\lim_{x \to a_i^-} f(x) = 0 \text{ and } \lim_{x \to a_i^+} f(x) = 1$$

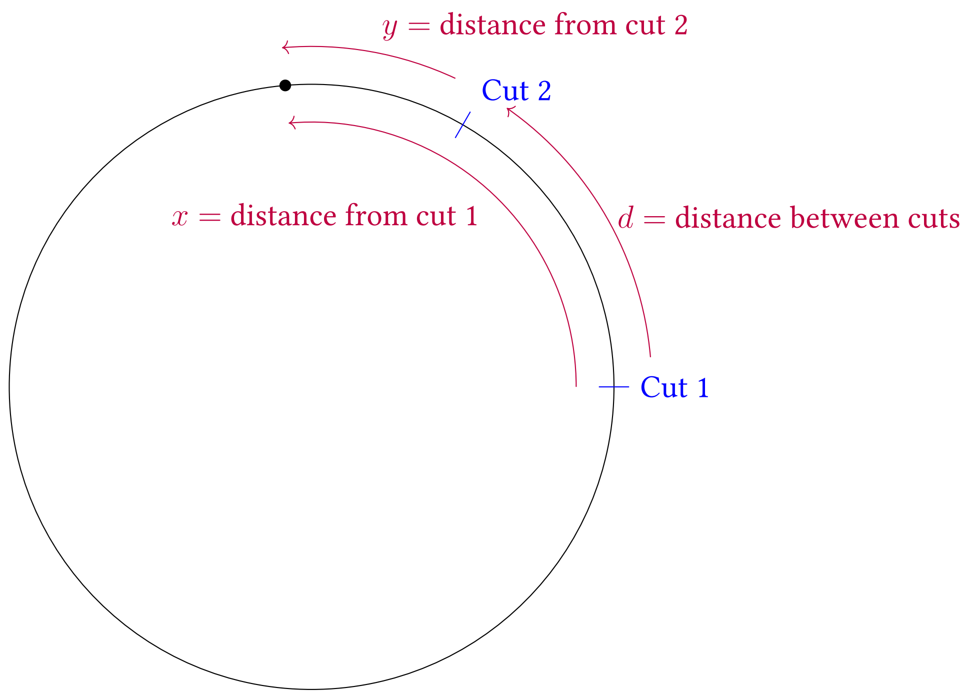
  for each i.

Example. Here is the graph of a continuous circle map.



Notice that we can have discontinuities as long as they "jump from 0 to 1 or back," and that the right endpoint matches the left endpoint.

Question. What happens if we cut the circle at a different point?



Expressing f in cut 1 through cut 2, call it g:

$$g(x) = f(x + d - \lfloor x + d \rfloor) - d - \lfloor f(x + d - \lfloor x + d \rfloor) - d \rfloor$$

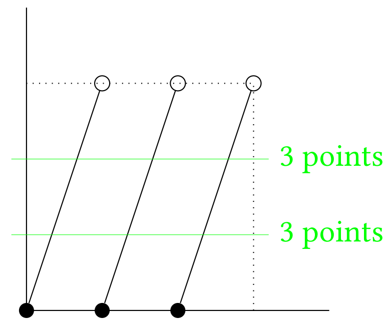
★**Exercise.** *Let $h(x) = x + d - \lfloor x + d \rfloor$. Show that $h(g(x)) = f(h(x))$, that h is a continuous circle map, that h is invertible, and that h^{-1} is a continuous circle map.*

★**Exercise.** *Assume f and g are continuous circle maps and that the function*

$$h(x) = f(x) + g(x) - \lfloor f(x) + g(x) \rfloor$$

has finitely many discontinuities. Show that h is a continuous circle map.

Lemma. *Assume f is a piecewise increasing continuous circle transformation. Then for every $y_1, y_2 \in [0,1)$, $\#\{x : f(x) = y_1\} = \#\{x : f(x) = y_2\}$. This common number is called the* <u>*degree*</u> *of f.*



Proof. Let $[a_1, a_2), [a_2, a_3), \ldots, [a_n, a_{n+1})$ denote the continuity domains of f, so $a_1 = 0$ and $a_{n+1} = 1$.

<u>Case 1.</u> $f(0) = 0$. Then $f([a_1, a_2)) = [0, 1)$ and $f(a_2) = 0$. For the same reason, $f([a_2, a_3)) = [0, 1)$ and so $f(a_3) = 0$. By induction, $f([a_i, a_{i+1})) = [0, 1)$ for each i. Hence each interval contributes at least one preimage for a given y by the intermediate value theorem. But since f is increasing, there is only one input in each of these intervals with output y. So, for each y, there are exactly n inputs x such that $f(x) = y$. We've shown that f has degree n.

<u>Case 2.</u> $f(0) \neq 0$. Let $\alpha = f(0)$. Then $f([a_1, a_2)) = [\alpha, 1)$ since f is increasing and can only have a jump discontinuity at a_2 from 1 to 0. Also, $f([a_n, a_{n+1})) = [0, \alpha)$. Finally, for each intermediate interval, we have $f([a_i, a_{i+1})) = [0, 1)$ for $1 < i < n$. Now, each y has $n - 1$ preimages. If $y \geq \alpha$ then it has a preimage in each of the first $n - 1$ intervals, and if $y < \alpha$, then it has a preimage in the last $n - 1$ intervals. □

★**Exercise.** *Show that there does not exist an increasing contraction on $[0, 1)$ which is a continuous circle map. Sub-exercise: Find a non-increasing contraction which is a continuous circle map.*

Definition. An <u>expanding circle map</u> is a piecewise differentiable circle transformation such that $\exists \lambda \in (1, \infty)$ such that $f'(x) \geq \lambda$ for all $x \in [0, 1)$.

★**Exercise.** *Show that if f is an expanding circle map, then f is not injective. [Hint: use the degree]*

Theorem. *If f is an expanding circle map of degree k, then f is conjugated to E_k by an invertible continuous circle transformation whose inverse is a continuous circle map.*