Day 5 (19 July 2024)

<u>Reminder</u> | $\bar{h}$ is defined only on $[x_1, x_0]$ and $\bar{h}(x_1) = y_1$ and $\bar{h}(x_0) = y_0$.

GOAL

<u>Def of conjugacy</u> | $h(x_n) = h(f^n(x_0))$
$$= g^n(h(x_0)) = g^n(y_0) = y_n$$

<u>Well-definedness & continuity</u> | Recall $h(x) := g^n(\bar{h}(f^{-n}(x)))$ is defined only on $x \in [x_{n+1}, x_n]$.

To verify continuity, we need to check the interval end points. We also need to define it at p:
$$h(p_1) := p_2.$$

Claim 1: h is well-defined.

↳ check at endpoints x_n: $x_n \in \underbrace{[x_{n+1}, x_n]}_{\mathrm{I}}$ and $x_n \in \underbrace{[x_n, x_{n-1}]}_{\mathrm{II}}$

↳ Def I:
$$\begin{aligned} h(x_n) &= g^n(\bar{h}(f^{-n}(x_n))) \\ &= g^n(\bar{h}(x_0)) \\ &= g^n(y_0) \\ &= y_n \end{aligned}$$

↳ Def II:
$$\begin{aligned} h(x_n) &= g^{n-1}(\bar{h}(f^{-(n-1)}(x_n))) \\ &= g^{n-1}(\bar{h}(x_1)) \\ &= g^{n-1}(y_1) \\ &= y_n \end{aligned}$$

Great! These are the same.

Verifying continuity at p_1: Suppose $x \to p_1^+$. Then if $x \in [x_{n+1}, x_n]$, then $n \to \infty$. Hence, since $h(x) \in [y_{n+1}, y_n]$, $h(x) \to p_2^+$.

<u>Exercise</u> | Construct the conjugacy using the same procedure using points less than p_1.

New topic!

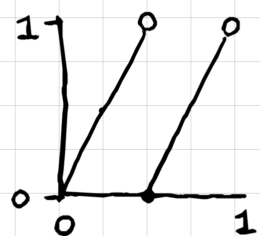
LINEAR EXPANDING MAPS on $[0,1)$

Def. $E_k : [0,1) \to [0,1)$

$$E_k(x) = \text{fractional part of } kx$$
$$= kx - \lfloor kx \rfloor$$

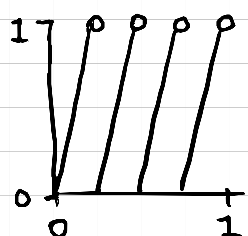
Graph of E_2:

$2x - \lfloor 2x \rfloor$

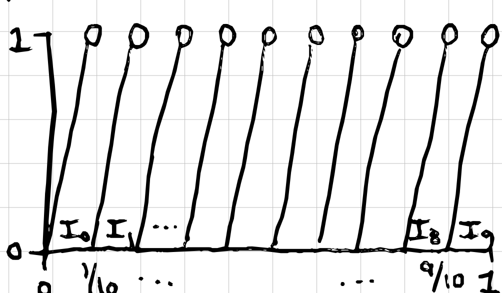


Graph of E_4:

$4x - \lfloor 4x \rfloor$



(Imprecise) Graph of E_{10}:



Digit expansion:
if
$x = 0.a_1a_2a_3\ldots$
then
$x \in I_{a_1}$

$$E_{10}(.a_1a_2a_3\ldots) = .a_2a_3\ldots$$
$$= a_1.a_2a_3\ldots - a_1$$

Main idea | The digit expansion describes the "itinerary" of x, telling us that

$$(E_k)^\ell(x) \in I_{a_{\ell+1}}.$$

Knowing this discrete itinerary tells us the points, i.e. what x is!

Connections with number theory |

- Rational numbers: $\mathbb{Q} := \{p/q : p \in \mathbb{Z},\ q \in \mathbb{N}\}$
  $= \{\text{eventually repeating decimals}\}$

Exercise | Show that

$$\mathbb{Q} := \{p/q : p \in \mathbb{Z},\ q \in \mathbb{N}\} = \{x : x \text{ is preperiodic for } E_{10}\}.$$

Ex: $E_{10}^3(.1\overline{32}) = E_{10}^2(.\overline{32}) = E_{10}(.\overline{23}) = .\overline{32}$.

Hints!

- $\Rightarrow$ use pigeonhole principle
- $\Leftarrow$ algebra w/ definition of E_{10}

Exercise | Generalize previous exercise to all E_k.

Exercise | How many periodic orbits are there for E_k with q as a period?