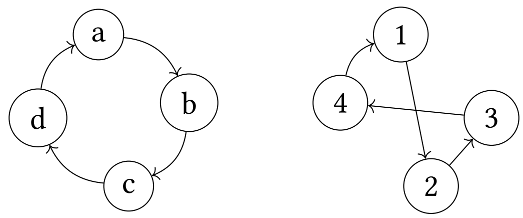
Goal. When are two dynamical systems "the same"?



Definition. *Let $f : X \to X$ and $g : Y \to Y$ be dynamical systems. f and g are said to be conjugated by a function $h : X \to Y$ if*

- *h is injective and surjective (aka bijective or invertible).*
- *For every $x \in X$, $h(f(x)) = g(h(x))$ (this is the conjugacy equation).*

If h is surjective (but not necessarily injective), we say that it is a semiconjugacy and that g is a factor of f.

We can capture the conjugacy equation in a diagram like so:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

which says that if we take a point $x \in X$, then the two "paths" to the bottom left are the same. I.e. in one direction, we apply f first and h second to get $h(f(x))$, and in the other direction, we apply h first and g second to get $g(h(x))$, and these are the same for a (semi)conjugacy.

Lemma. *If h is a conjugacy between f and g, then $\forall n \in \mathbb{N}$ and $\forall x \in X$,*

$$h(f^n(x)) = g^n(h(x))$$

Proof. By induction.

Base case. For $n = 1$, we need to show:

$$h(f(x)) = g(h(x))$$

which we already know (the conjugacy equation).

Inductive step. Assume $h(f^n(x)) = g^n(h(x))$ for some n. Then,

$$h(f^{n+1}(x)) = h(f(f^n(x))) = g(h(f^n(x))) = g(g^n(h(x))) = g^{n+1}(h(x))$$

So, the claim is true for all $n \in \mathbb{N}$. $\square$

Remark: we can think of this proof as follows:

$$\begin{aligned} &h(f(f(f(\cdots f(x)\cdots)))) \\ =\ &g(h(f(f(\cdots f(x)\cdots)))) \\ =\ &g(g(h(f(\cdots f(x)\cdots)))) \\ &\vdots \end{aligned}$$

Theorem. *If h is a conjugacy between f and g, and p is f-periodic with period q, then $h(p)$ is g-periodic with period q.*

Proof. Since p is f-periodic with period q, $f^q(p) = p$, and $f^n(p) \neq p$ when $1 \leq n \leq q-1$.

Then $g^q(h(p)) = h(f^q(p)) = h(p)$. So $h(p)$ is periodic with period at most q. We need to show that this is the smallest exponent. For $1 \leq n \leq q-1$, we have $g^n(h(p)) = h(f^n(p)) \neq h(p)$, where the inequality comes from the fact that $f^n(p) \neq p$ and that h is injective. $\square$

★**Exercise.** *What are the possible periods if h is a semiconjugacy?*

Example. *The dynamical systems on $\mathbb{R}$ defined by*

$$f(x) = \frac{1}{2}x \text{ and } g(x) = \frac{1}{3}x$$

are conjugate.

In other words, there is some function h with $h(f(x)) = g(h(x))$. I.e.

$$h\left(\frac{1}{2}x\right) = \frac{1}{3}h(x)$$

We'll make the guess that h has the form $h(x) = x^\gamma$ for some $\gamma \in (0, \infty)$. Then, the above equation becomes

$$\begin{aligned} \left(\frac{1}{2}x\right)^\gamma &= \frac{1}{3}x^\gamma \\ \left(\frac{1}{2}\right)^\gamma x^\gamma &= \frac{1}{3}x^\gamma \\ \left(\frac{1}{2}\right)^\gamma &= \frac{1}{3} \\ \ln\left(\left(\frac{1}{2}\right)^\gamma\right) &= \ln\left(\frac{1}{3}\right) \\ \gamma \ln\left(\frac{1}{2}\right) &= \ln\left(\frac{1}{3}\right) \\ -\gamma \ln(2) &= -\ln(3) \\ \gamma &= \frac{\ln(3)}{\ln(2)} \end{aligned}$$

So we'd like to take the conjugacy to be $h(x) = x^{\ln(3)/\ln(2)}$. But we can't raise negative numbers to non-integer powers, so we pick

$$h(x) = \begin{cases} x^{\ln(3)/\ln(2)} & \text{if } x \geq 0 \\ -|x|^{\ln(3)/\ln(2)} & \text{if } x < 0 \end{cases}$$

★**Exercise.** *Check that this is a conjugacy.*

Lemma. *If $f : \mathbb{R} \to \mathbb{R}$ has $f'(x) > 0$ for all $x \in \mathbb{R}$ and $a < b$, then $f(a) < f(b)$.*

★**Exercise.** *Prove this lemma using the mean value theorem.*

Theorem. *If f and g are contractions on $\mathbb{R}$, and $f'(x), g'(x) > 0$ for all $x \in \mathbb{R}$, then they are conjugated by a continuous function h.*

In other words, we're saying: "Up to conjugacy, there is only 1 increasing contraction on $\mathbb{R}$."

Question. When are both h and h^{-1} differentiable? [This is hard; think of an answer, but do not try to prove anything]

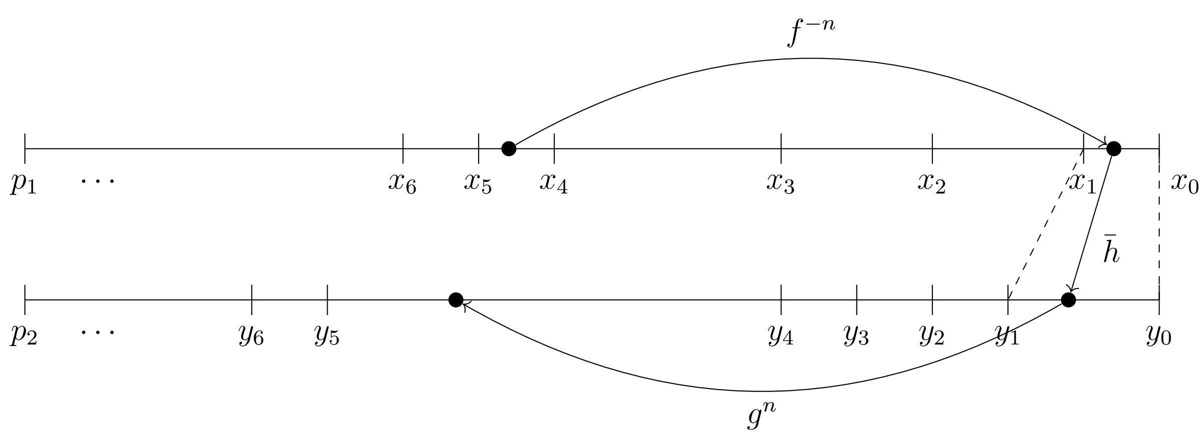
Proof. Let p_1 be the (unique) fixed point of f and p_2 be the fixed point of g.

Choose $x_0 = p_1 + 1$ and $y_0 = p_2 + 1$. Define $x_n = f^n(x_0)$ and $y_n = g^n(y_0)$. Let $\bar{h}(x)$ be the linear function passing through the points (x_1, y_1) and (x_0, y_0). Define $h(x)$ on the interval $[x_{n+1}, x_n]$ by $h(x) = g^n(\bar{h}(f^{-n}(x)))$.

We need to check that this is well-defined and that it satisfies the conjugacy equation. First let's show that it conjugates. Let $x > p_1$, so that $x \in [x_{n+1}, x_n]$ for some $n \in \mathbb{Z}$. Then $f(x) \in [x_{n+2}, x_{n+1}]$. Then,

$$h(f(x)) = g^{n+1}(\bar{h}(f^{-n-1}(f(x)))) = g^{n+1}(\bar{h}(f^{-n}(x))) = g(g^n(\bar{h}(f^{-n}(x)))) = g(h(x))$$

□



★**Exercise.** *Show that h above is well-defined.*