Day 2 (16 July 2024)

GOAL: Prove the following

THM Let $f: X \to X$ be a dynamical system on a finite set X. Then every point $x \in X$ is preperiodic and X has a periodic orbit.

REMARK: Provable facts usually fall into 3 types — lemmas, propositions, and theorems. Theorems are the results we "care about" and lemmas are facts we need to prove other facts (but which may not be as interesting in its own right.).

LEMMA $x \in X$ is pre-periodic for T if and only if $\exists k \neq l$ such that $T^k(x) = T^l(x)$.

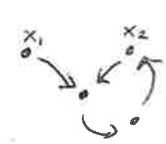
PROOF First, assume that x is pre-periodic. Then, there exists k such that $T^k(x)$ is periodic. Since $T^k(x)$ is periodic, there exists a q such that $T^q(T^k(x)) = T^k(x)$. Setting $l = q + k$ proves one direction of the lemma.

Now, assume $\exists k \neq l$ such that $T^k(x) = T^l(x)$. Without loss of generality, assume $l > k$. Set $q = l - k \in \mathbb{N}$. Therefore, $T^q(T^k(x)) = T^{q+k}(x) = T^l(x) = T^k(x)$. So, $T^k(x)$ is periodic, and so x is pre-periodic. $\square$

DEFINITION Let $f: X \to Y$ be a function.

- f is called injective or one-to-one if whenever $x_1 \neq x_2$ with $x_1, x_2 \in X$, then $f(x_1) \neq f(x_2)$.
- f is called surjective or onto if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

REMARK: In the dynamical system pictures we've drawn, situations like this



are not injective, ~~It~~ since $T(x_1) = T(x_2)$ but $x_1 \neq x_2$. It is also not surjective since $T(x) \neq x_1$ for any $x \in X$.

LEMMA Let X and Y be finite sets, and $f: X \to Y$ be a function.

- If f is injective, then $\#(X) = \#(f(X)) \leq \#(Y)$, where $f(X) = \{f(x) \in Y : x \in X\}$ [the "range"].
- If f is surjective, then $\#(X) \geq \#(f(X)) = \#(Y)$

PROOF Omitted.

LEMMA (Pidgeonhole Principle). If $\#(X) > \#(Y)$, then any $f: X \to Y$ is not injective.

PROOF This is the contrapositive of the previous.

DISCUSSION (contrapositive): If we have a ~~statement~~ statement of the form "If A then B", then it is not necessarily true to reverse it ("If B then A"). However, it does remain true if we negate both pieces as well: "If not B then not A." For example, "If Fred is a dog, then Fred is a mammal" is true. Flipping it is false — "If Fred is a mammal, then Fred is a dog" — since Fred could be a cat. But the contrapositive is true: "If Fred is not a mammal, then Fred is not a dog."

PROOF OF GOAL THEOREM Pick any $x \in X$. Let N denote the number of points in X. Define the function $g: \{0, 1, 2, \dots, N\} \to X$ by $g(n) = f^n(x)$. Since $\#(\{0, 1, \dots, N\}) = N+1$, g is not injective by the pigeonhole principle. Therefore, there exists $k \neq \ell$ such that $g(k) = g(\ell)$. That is, $f^k(x) = f^\ell(x)$. By the first lemma today, x is pre-periodic.

EXAMPLE A linear contraction on $\mathbb{R}$ is a function $T(x) = \lambda x + c$ for some $c \in \mathbb{R}$ and $\lambda \in (0,1)$. E.g. for $\lambda = \frac{1}{2}$ and $c = 0$, we have $T(x) = \frac{1}{2}x$.

EXERCISE For any linear contraction T on $\mathbb{R}$, show that there exists a unique fixed point p and if $x \in \mathbb{R}$ then $\lim_{n\to\infty} T^n(x) = p$.

EXAMPLE For $T(x) = \frac{1}{2}x$ as above, $p = 0$ is the only fixed point and for any $x \in \mathbb{R}$,

$$T^n(x) = \frac{1}{2^n}x \quad \text{so} \quad \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} \frac{x}{2^n} = 0 = p.$$

DEFINITION A contraction on $\mathbb{R}$ is a dynamical system $T: \mathbb{R} \to \mathbb{R}$ such that there exists some $\lambda \in (0,1)$ satisfying $|T'(x)| \leq \lambda$ for all $x \in \mathbb{R}$.

THEOREM (Contraction Mapping Principle). If T is a contraction on $\mathbb{R}$, then T has a unique fixed point p and for all $x \in \mathbb{R}$, $\lim_{n\to\infty} |T^n(x) - p| = 0$

(i.e. $\lim_{n\to\infty} T^n(x) = p$)