Day 1 (15 July 2024)

Goal: Give precise and provable structures to patterns and models.

Two key aspects:

- Make precise definitions (using set theory)
- Prove theorems about interactions

Ex. Def: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{R} = \{\text{real numbers}\}$ = numbers on number line

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Notation:

- $3 \in \mathbb{Z}$ means "3 is an element of $\mathbb{Z}$"
- $2.5 \notin \mathbb{Z}$ means "2.5 is not an element of $\mathbb{Z}$"
- If A, B are sets,
  - $A \cup B$ = things in A and/or B
    $= \{x : x \in A \text{ or } x \in B\}$

    thing in set   condition(s) on thing in set
  - $A \cap B = \{x : x \in A \text{ and } x \in B\}$

"Thm": "Division with remainder makes sense for $\mathbb{N}$"

Theorem: If $p, q \in \mathbb{N}$, then there exists a pair of non-negative integers n, r such that $0 \leq r < q$ and $p = qn + r$.

Proof: (idea: to find n we want the largest multiple of q that's less than or equal to p. r is then what is leftover)

Consider the set $A = \{n : n \in \mathbb{N}_0 \text{ and } nq \leq p\}$. ($\mathbb{N}_0$: $\mathbb{N} \cup \{0\}$)

First, $A \neq \emptyset$, since $0 \in A$. Second, A is bounded

($\emptyset$: empty set)

there exists
($\exists M$ such that if $nq \in A$, then $nq \le M$). Why?
Since $q \in \mathbb{N}$, $q \ge 1$, and so $(p+1)q \ge p+1 > p$,
and so any number $\ge p+1$ is not in A. ($M = p+1$.)
We use the well-ordered property: Given any bounded subset of natural numbers, that set has a maximum.
Let n be the maximal element of A. Define $r := p - qn$.

problem session: It remains to show that $0 \le r < q$. We use that $n \in A$ and $n+1 \notin A$.

(solved during groupwork)

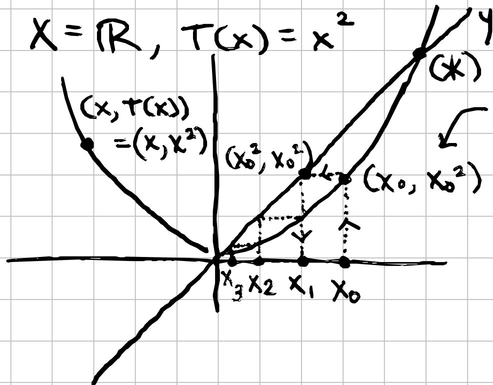
Dynamical systems

- "Def": A dynamical system is an evolution rule in a closed environment.
- Def: A dynamical system is a set X (closed environment) and a function $f : X \to X$ (evolution rule)
  inputs (domain) outputs (codomain)
- Interpretation:
  - x_0 = starting position ($x_0 \in X$)
  - x_1 = position at time 1 = $f(x_0)$
  - $x_2 = f(x_1) = f(f(x_0))$
  - $x_3 = f(x_2) = f(f(x_1)) = f(f(f(x_0)))$
  - $x_n = f(x_{n-1}) = f^n(x_0)$ ← NOT $(f(x_0))^n$
    $= f(f(f(\dots(f(x_0))$ n times
- Def: If $x_0 \in X$, the forward orbit of x_0 is the set $\mathcal{O}(x_0) = \{f^n(x_0) : n \ge 0\}$. (Here, $f^0(x_0) = x_0$ by definition.)

- Properties of iteration
  - $f^n(f^m(x_0)) = f^{n+m}(x_0)$
  - $(f^n)^k(x_0) = f^{nk}(x_0)$

Remark: For now we are only considering non-negative iterations since we have not said anything about invertibility of our dynamical system.

Ex. $X = \mathbb{R}$, $T(x) = x^2$



To iterate, we want to turn our y-coordinate into our new x-coordinate

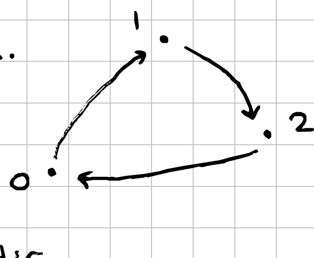
Remark: If x_0 starts further right than (*) which is at $(1,1)$, then iterating sends us to infinity!

- Def: A point x is <u>fixed</u> by T if $T(x) = x$.

What are the fixed points of $T(x) = x^2$? $x = 0, 1$
prove in problem session

Ex. $X = \{0, 1, 2\}$, $T: X \to X$ where $T(0) = 1$, $T(1) = 2$, $T(2) = 0$

- Def: x is <u>periodic</u> if there exists $q \in \mathbb{N}$ such that $T^q(x) = x$.
- Every point of $T(x)$ is periodic. Because the loop is length 3, we call it 3-periodic.
- Def: A point is <u>preperiodic</u> if $\exists r \in \mathbb{N}$ such that $f^r(x)$ is periodic.
  there exists



- Theorem: If X is a finite set and $T: X \to X$ is a dynamical system, then T has a periodic orbit, and every orbit is either periodic or preperiodic.

problem session: If x is periodic, then $\exists$ unique $q \in \mathbb{N}$ such that $\{kq : k \in \mathbb{N}\} = \{n \in \mathbb{N} : T^n(x) = x\}$.

↳ proof technique: (set equality)
$A = B$ if $A \subset B$ and $B \subset A$

problem session: (Induction) If x is fixed, then $T^n(x) = x$ for all $x \in X$.