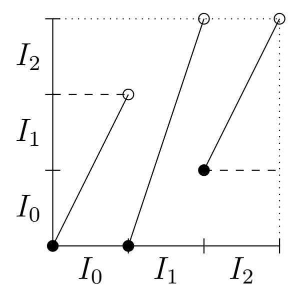
Example. Consider the function

$$f(x) = \begin{cases} 2x & 0 \le x < 1/3 \\ 3x - 1 & 1/3 \le x < 2/3 \\ 2x - 1 & 2/3 \le x < 1 \end{cases}$$

with graph:



Note: this is not a continuous circle map. It still has 1-sided inverses. Since $f(I_0) = f([0, 1/3)) = [0, 2/3) = I_0 \cup I_1$, we get an inverse $g_0 : [0, 2/3) \to [0, 1/3)$ given by

$$g_0(x) = \frac{1}{2}x$$

Similarly, we can compute $g_1 : [0, 1) \to [1/3, 2/3)$ and $g_2 : [1/3, 1) \to [2/3, 1)$:

$$g_1(x) = \frac{1}{3}x + \frac{1}{3}$$
$$g_2(x) = \frac{1}{2}x + \frac{1}{2}$$

We now have to be careful when building compositions

$$g_{a_1} \circ g_{a_2} \circ \cdots \circ g_{a_\ell}$$

since we need to make sure the output of $g_{a_{i+1}}$ is an input of g_{a_i}.

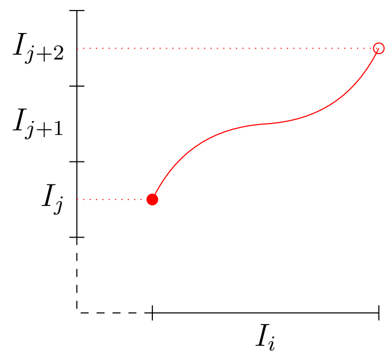
Definition. Let $f : [0, 1) \to [0, 1)$ be a piecewise-increasing interval map, which is continuous on a family of subintervals $[d_i, d_{i+1})$ with $0 = d_0 < d_1 < \cdots < d_k = 1$.

f is said to have the Markov property relative to $\{d_0, \ldots, d_k\}$ if for all i, $f([d_i, d_{i+1}))$ is a union of other intervals of the same type.

In other words, for every i, there exists a set $B_i \subseteq \{0, \ldots, k-1\}$ such that $f(I_i) = \bigcup_{j \in B_i} I_j$.

Remark. Note that the $[d_i, d_{i+1})$ don't have to be full continuity domains. We're allowed to "chop up" our continuity domains into even smaller pieces if it allows us to fulfill the Markov property.

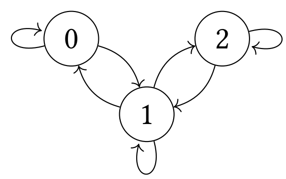
Example (Non-Markov). A function that doesn't have the Markov property is one with a "partial crossing." E.g.



Definition. A (directed) graph is a collection of vertices (nodes) and (directed) edges (arrows).

The graph associated to a Markov system has its vertices equal to the set $\{0, \ldots, k-1\}$ (representing coding intervals), and an edge from i to j if and only if $I_j \subseteq f(I_i)$.

Example. For the example above, the associated graph would be:



Definition. If $w = a_1 a_2 \ldots a_\ell$, w is called an <u>admissible word</u> if $\forall i = 1, \ldots, \ell - 1$, there is an edge from a_i to a_{i+1}. Intuitively: "following the word moves along the graph."

Example. For the previous example, the following are admissible words:

$$0101, 000000, 11012$$

and the following are not admissible:

$$0211, 12120$$

In particular, any word with a 2 after a 0 or a 0 after a 2 is not admissible.

Remark. By construction, every code of an orbit yields an admissible word. The reverse is also true. Indeed:

Lemma. *For an admissible word $w = a_1 a_2 \ldots a_\ell$, define*

$$I_w = g_{a_1}\left(g_{a_2}\left(\cdots\left(g_{a_\ell}\left(\bigsqcup_{j \in B_{a_\ell}} I_j\right)\right)\cdots\right)\right)$$

Then I_w is well-defined, $\text{length}(I_w) \le \lambda^{-\ell}$ *(if f is expanding), and*

$$[0,1) = \bigsqcup_{\substack{w \text{ admissible} \\ w \text{ of length } \ell}} I_w$$

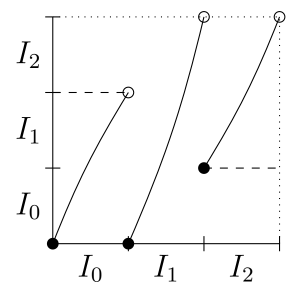
Recall. The notation

$$\bigsqcup_{j\in\{b_1,\ldots,b_m\}} I_j = I_{b_1} \cup \cdots \cup I_{b_n}$$

and that there are no overlaps, i.e. $I_{b_i} \cap I_{b_j} = \emptyset$ for $i \neq j$.

Theorem. *If f_1 and f_2 are piecewise increasing, expanding, Markov maps with the same associated graph (labeled the same), then there is a continuous conjugacy between f_1 and f_2.*

Example. Note that having the same associated graphs doesn't mean $f_1 = f_2$. E.g. f_1 could be our example from above, while f_2 could have graph:



Remark. Some subtleties arise:

- Forbidden terminating digits occur when a right hand endpoint is fixed.
- f increasing implies that each g_i is increasing, which implies that the intervals I_w are listed "in order" as before.
- f is a continuous circle map if and only if every branch is "full" if and only if the associated graph is complete (it has every possible edge).

★**Exercise.** *Find the coding intervals of lengths 2 and 3 for the explicit example here.*