Theorem

Let α be the root of a polynomial

$f(x) = b_d x^d + \dots + b_1 x + b_0 = 0,$

where each $b_i \in \mathbb{Z}$ and the polynomial has no rational roots. Then $\exists c > 0$ s.t. for any $\frac{p}{q} \in \mathbb{Q}$ in reduced form

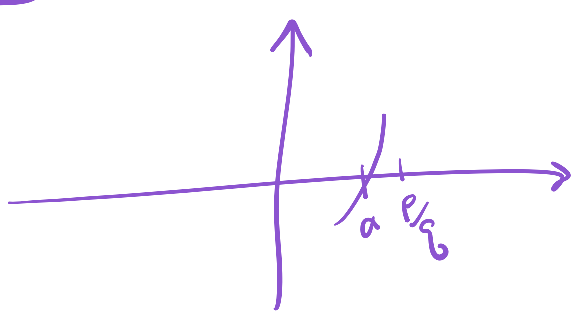
$$|q\alpha - p| \geq \frac{c}{q^{d-1}}$$

$$\left(\text{ie } \left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d} \right)$$

Pf We know that $f(\alpha) = 0$

idea:

look at $\left|f\left(\frac{p}{q}\right)\right|$



$$\left|f\left(\frac{p}{q}\right)\right| = \left|b_d\left(\frac{p}{q}\right)^d + \dots + b\,\frac{p}{q} + b_0\right| \qquad \star$$

combine into one fraction

$$= \left|\frac{\overbrace{b_d p^d + b_{d-1} q p^{d-1} + \dots + b_1 p q^{d-1} + b_0 q^d}^{\in \mathbb{Z}}}{q^d}\right| \geq \frac{1}{q^d}$$

estimate $f\left(\frac{p}{q}\right)$ another way?

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\alpha)$$

by the mean value theorem,

$\exists\, z \in \left(\alpha, \frac{p}{q}\right)$ s.t.

$$\left|f\left(\frac{p}{q}\right) - f(\alpha)\right| = f'(z)\left(\frac{p}{q} - \alpha\right)$$

$f'(z) \leq c$ when z is close to α.

for some c.

so we have

$$\left|f\left(\tfrac{p}{q}\right)\right| = \left|f\left(\tfrac{p}{q}\right) - f(\alpha)\right| = |f'(z)|\left|\tfrac{p}{q} - \alpha\right| \leq c\left|\tfrac{p}{q} - \alpha\right| \qquad (\star\star)$$

$\star$ and $\star\star$ together imply

$$\left|\frac{p}{q} - \alpha\right| \geq \frac{1/c}{q^d}$$

Series of excersices for today

$\boxed{1}$ Consider an expression of the form

$$f(x) = \frac{ax+b}{cx+d} \qquad a,b,c,d \in \mathbb{Z}.$$ Show that

$$\frac{1}{n+f(x)}$$ is an expression of the same form.

$\boxed{2}$ If α has periodic continued fraction expansion then

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots \cfrac{1}{n_k + \alpha}}}$$ for some k.

$\boxed{3}$ Combine 1 & 2 to show that any fraction with periodic digits is a quadratic irrational (ie the root of a quadratic eqn with integer coefficients)

$\boxed{0}$ Show that if $\alpha = \dfrac{1}{a+\alpha}$

α is a quadratic irrational.