Theorem

Let \( a \) be the root of a polynomial

\[ f(x) = b_d x^d + \ldots + b_1 x + b_0 = 0, \]

where each \( b_i \in \mathbb{Z} \) and the polynomial has no rational roots. Then \( \exists C > 0 \) s.t. for any \( \frac{p}{q} \in \mathbb{Q} \) in reduced form

\[ \left| q \alpha - p \right| \geq \frac{C}{q^{d-1}}. \]

\[ \left| \alpha - \frac{p}{q} \right| \geq \frac{C}{q^d} \]

Proof: We know that \( f(\alpha) = 0 \)

idea:

look at \( f\left( \frac{p}{q} \right) \)
\[ f(\frac{p}{q}) = b_d (\frac{p}{q})^d + \ldots + b_0 \]

combine into one fraction

\[ = \frac{b_d p^d + b_{d-1} p^{d-1} + \ldots + b_0}{q^d} \]

\[ \geq \frac{1}{q^d} \]

estimate \( f(\frac{p}{q}) \) another way?

\[ f(\frac{p}{q}) = f(\frac{p}{q}) - f(\alpha) \]

by the mean value theorem,

\[ \exists \ z \in (a, \frac{p}{q}) \ such \ that \]

\[ |f(\frac{p}{q}) - f(\alpha)| = f'(z)(\frac{p}{q} - \alpha) \]

\[ f'(z) \leq c \ when \ z \ is \ close \ to \ \alpha \]

for some \( c \).
So we have

\[
|f(\frac{p}{q})| = |f(\frac{p}{q}) - f(a)| = |f'(c)| |\frac{p}{q} - a| \\
\leq c |\frac{p}{q} - a|
\]

and \(\bigcirc\) and \(\bigcirc\bigcirc\) together imply

\[
|\frac{p}{q} - a| \geq \frac{\gamma c}{q^d}
\]
Series of exercises for today

1) Consider an expression of the form
   \[ f(x) = \frac{ax + b}{cx + d} \quad a, b, c, d \in \mathbb{Z}. \]
   Show that
   \[ \frac{1}{n + f(x)} \]
   is an expression of the same form.

2) If \( h \) has periodic continued fraction expansion then
   \[ h = \frac{1}{n_1 + \frac{1}{n_2 + \cdots + \frac{1}{n_k + \cdots}}} \]
   for some \( k \).

3) Combine 1 & 2 to show that any fraction with periodic digits is a quadratic irrational
   (i.e., the root of a quadratic equation with integer coefficients)
Show that if \( \alpha = \frac{1}{a + \alpha} \), \( \alpha \) is a quadratic irrational.