Looking at distribution of finite orbits

Recall \[ \text{Thm}\] Let \( R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be a rotation by an angle \( \alpha \), then either

(a) every orbit has period \( q \) where \( \alpha = \frac{p}{q} \) in reduced form

(b) every orbit is dense \( (\alpha \not\in \mathbb{Q}) \)

How to build rational rotations with orbits being less and less sparse.

Naïve approach

\[ \alpha = \frac{1}{n} \] not helpful.

Issue: the limiting rotation \( n \to \infty \) is \( \alpha = 0 \) which is uninteresting.

Want: something irrational in limit?
Another more interesting example:

Start with $R_{13}$

The gap between 0 and $3a$ is

$p$ so

$\beta = 1 - 3a$

$5\beta = a$

Now can solve!

\[
\frac{a}{5} = 1 - 3a
\]

$3a + \frac{a}{5} = 1$

$a(3 + \frac{1}{5}) = 1$

$q = \frac{1}{3 + \frac{1}{5}} = \frac{5}{16}$

First step of a continued fraction

Check
3 steps to come back to wedge and each step is like $\beta$

1st return map to $[0, \alpha)$:

If $x \in [0, \alpha)$ define

$$F(x) = 1st \ time \ the \ forward \ orbit \ of \ x \ revisits \ [0, \alpha).$$

In the example, if the wedge is stretched into a circle, the 1st return map is another rotation by angle $\frac{1}{5}$.

**Exercise 1**

Find equations explaining why we can think of the first return $a$

$$a = \frac{1}{x_1 + \frac{1}{a_2}}$$

to the interval $[0, \alpha)$ as a rotation by $\frac{1}{a_2}$.

We can iterate this procedure.

Consider 1st return if this regime

Go around 4 times to get periodic

$$\frac{1}{5} \to \frac{1}{5 + \frac{1}{4}} \ \text{with} \ a \ approx ...$$
\[
q = \frac{1}{3 + \frac{1}{5 + \frac{1}{q}}}
\]

**First return maps**

**Def** let \( f : x \mapsto x \) be a dynamical system and \( A \subset X \) (our wedge before)

The first return dynamics to \( A \) is the dynamical system

\[
F : A \to A \quad \text{such that}
\]

\[
F(x) = \text{the first positive iterate } f^k(x) \text{ such that } f^k(x) \in A.
\]

**Exercise 2**

Prove that the first return map \( q : R^c X \) for the interval of \([0, q)\)

is conjugated to the rotation by angle \( \frac{1}{a} - \left\lfloor \frac{1}{a} \right\rfloor \)

**Exercise 0**

Draw some orbits and connect to the lecture \( q = \frac{1}{2 + \frac{1}{2 + \frac{1}{q}}} = \frac{3}{7} \)

and \( q = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{q}}}}} = \frac{5}{14} \)