Overview of what we've learned:

**Main goal:** Understand rotation dynamics

$$R_{\alpha}: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

$$R_{\alpha}([x]) = [x + \alpha]$$

**Idle:** Understand orbits!

$$\Theta_{\pm}(\mathbb{Z}) = \left\{ R_{\alpha}^k([x]): k \in \mathbb{Z} \right\}$$

$$= \left\{ [x + n\alpha]: n \in \mathbb{Z} \right\}$$
First week: Rational rotations thought about:

\[ \frac{TR}{2\pi \mathbb{Z}} \quad \alpha = \frac{p}{q} \quad \text{in reduced form} \]

Denominator gives the period, i.e. \( q = \text{period} \)

All orbits evenly spaced by \( \frac{1}{q} \).

This week: Building structures to understand irrational rotations

I) Understand \( \mathbb{R} \mathbb{Z} \) and \( \mathbb{R} \) have group structures

II) Connected dynamics and group structure
By thm
\[ p^{-1}(\Theta_\pm([0])) \text{ is a subgroup of } \mathbb{R}. \]

III) Classifying subgroups of \( \mathbb{R} \).

If Thm (Main thm)
\( G \subset \mathbb{R} \) is a subgroup, then either

(a) \( G = c\mathbb{Z} \) for some \( c > 0 \).

OR

(b) \( G \) is dense in \( \mathbb{R} \).

Last time

Thm) If \( G \) is not dense (i.e., not in case (b))
then \( \exists c > 0 \) s.t. \((0,c) \cap G = \emptyset\).
Sketch & Proof of Main Theorem

Claim: \( G \) has a smallest positive element.

 gaps between elements

\( \Rightarrow \) small element of group since can add/subtract them

Claim: all gaps are of size \( x \)

ie smallest element of \( G \) is \( x \), then all of \( G \) are of the form \( x \cdot n \), \( n \in \mathbb{Z} \).
$XZ \subseteq G$ and $G \subseteq XZ$ by lemmas/exercises from yesterday.