Overview of what we've learned:

main goal: understand rotation dynamics

$$R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

$$R_\alpha([x]) = [x+\alpha]$$

idea: understand orbits!

$$\mathcal{O}_{\pm}([x]) = \{R_\alpha^k([x]) : k \in \mathbb{Z}\}$$

$$= \{[x+n\alpha] : n \in \mathbb{Z}\}$$

First week | Rational rotations

thought about:

$\mathbb{R}/2\pi\mathbb{Z}$ $\alpha = \frac{p}{q}$ in reduced form

denominator gives the period
ie q = period

All orbits evenly spaced by $\frac{1}{q}$.

This week | Building structures to understand irrational rotations

I) Understand $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$ have group structures

II) Connected dynamics and group structure

↳ By thm

$p^{-1}(\mathcal{O}_{\pm}([0]))$ is a subgroup of $\mathbb{R}$.

III) Classifying subgroups of $\mathbb{R}$.

Thm (main thm)

if $G \subset \mathbb{R}$ is a subgroup, then either

(a) $G = c\mathbb{Z}$ for some $c \geq 0$.

OR

(b) G is dense in $\mathbb{R}$.

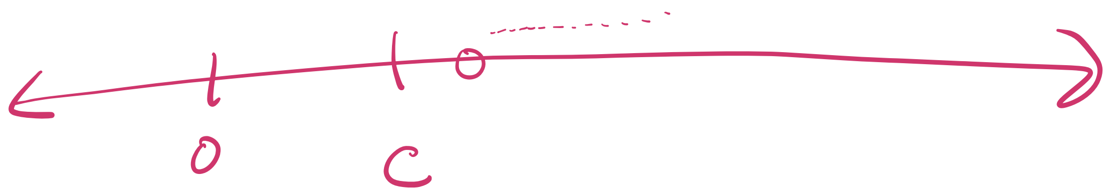
Last time

Thm) If G is not dense (ie not in case b)

then $\exists\, c > 0$ s.t. $(0, c) \cap G = \emptyset$.

Sketch of Proof of main theorem

Claim: G has a smallest positive element.



gaps between elements

$\Rightarrow$ small element of group since can add/subtract them ↯

Claim: all gaps are of size x

ie smallest element of G is x, then all elements of G are of the form $x \cdot n$, $n \in \mathbb{Z}$.

$x\mathbb{Z} \subset G$ and $G \subset x\mathbb{Z}$ by lemmas/exercises from yesterday.