$\underline{\text{Ex}}$ $(\mathbb{R}/\mathbb{Z}, +)$ is a group.

① The identity element is $[0]_{\mathbb{Z}}$

since $[0]_{\mathbb{Z}} + [x]_{\mathbb{Z}} = [x+0]_{\mathbb{Z}}$

$= [x]_{\mathbb{Z}}$

② Associativity

$$([x]_{\mathbb{Z}} + [y]_{\mathbb{Z}}) + [z]_{\mathbb{Z}}$$

$$= [x+y]_{\mathbb{Z}} + [z]_{\mathbb{Z}}$$

$$= [(x+y)+z]_{\mathbb{Z}} = [x+(y+z)]_{\mathbb{Z}}$$

$$= [x]_{\mathbb{Z}} + [y+z]_{\mathbb{Z}}$$

$$= [x]_{\mathbb{Z}} + ([y]_{\mathbb{Z}} + [z]_{\mathbb{Z}})$$

:)

exercise: Show that

this plus is different than this plus sign

$$[x]_{\mathbb{Z}} \boxplus [y]_{\mathbb{Z}} = [x+y]_{\mathbb{Z}}$$

is well defined.

choose

$$[v]_{\mathbb{Z}} = [x]_{\mathbb{Z}}, \quad [w]_{\mathbb{Z}} = [y]_{\mathbb{Z}}$$

show that

$$[v+w]_{\mathbb{Z}} = [x+y]_{\mathbb{Z}}$$

③ Inverses:

$$-[x]_{\mathbb{Z}} = [-x]_{\mathbb{Z}}$$

$$-[x]_{\mathbb{Z}} + [x]_{\mathbb{Z}} = [-x]_{\mathbb{Z}} + [x]_{\mathbb{Z}}$$

$$= [-x+x]_{\mathbb{Z}} = [0]_{\mathbb{Z}} \quad ✓$$

Things that $\mathbb{R}/\mathbb{Z}$ doesn't have:

- Notation of $\leq$ or $\geq$
- positive/negative
- multiplication (not on equivalence class)
- division
- powers/exponents
- square roots

Recall: $p: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

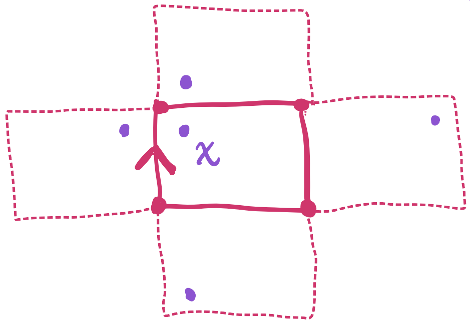
$$p(x) = [x]_{\mathbb{Z}}$$

Recall fundamental domain

A fundamental domain of $\mathbb{R}$ is "a smallest set containing exactly one element of every equivalence class"

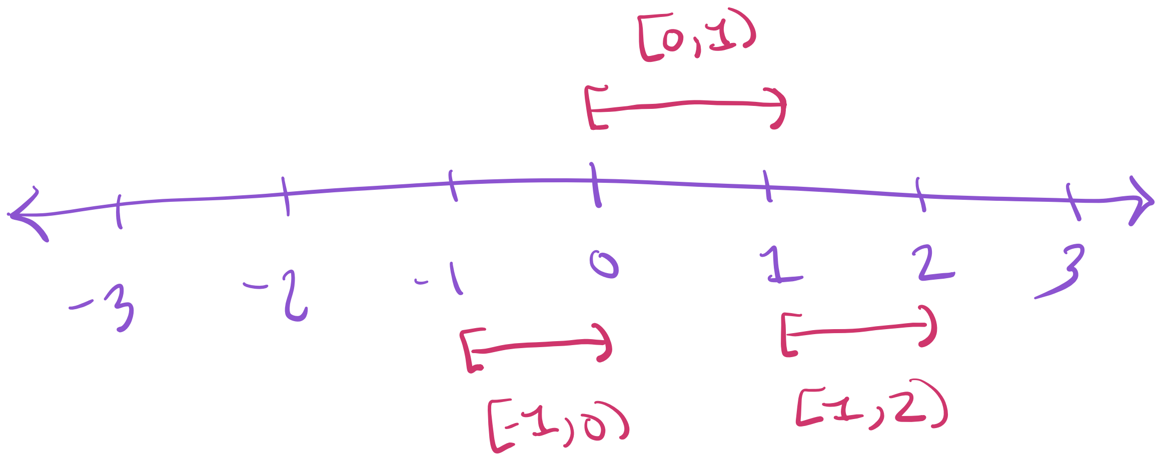
not unique

$\bullet = \mathcal{O}_{\pm}(x)$



"group action" is reflections across lines/borders of fundamental domain"

different fundamental domains



$[0,1)$ is a fundamental domain of $\mathbb{R}/\mathbb{Z}$

since $x - \lfloor x \rfloor \in [0,1)$ and

$$[x - \lfloor x \rfloor]_{\mathbb{Z}} = [x]_{\mathbb{Z}}$$

Exercise) Show that $\forall x$, $[x]_{\mathbb{Z}}$ intersects $[0,1)$ in at most one point.

(Hint: show that if $y \in [0,1)$ and $n \in \mathbb{Z}$ then either $y+n \geq 1$ or $y+n < 0$)

Recall

$\lfloor x \rfloor :=$ the largest integer $\leq x$

eg $\lfloor 5.3 \rfloor = 5$.

Notice any half open interval of length 1 is an equivalence class

Exercise) Show that any half open interval of length 1 is a fundamental domain.

Theorem if $F \subset \mathbb{R}$ is a fundamental domain, then

$$p|_F : F \to \mathbb{R}/\mathbb{Z}$$

"p restricted to F"
(only allows inputs from F)

this function is 1-1 and onto!

(ie it is invertible!)

Note the theorem is a technical way of establishing "dictionaries" between fundamental domains and $\mathbb{R}/\mathbb{Z}$

ie each fundamental domain wraps around the circle exactly once.

pf of thm

onto: We need to show that for any $[x]_Z$ there exist $y \in F$ s.t. $p(y) = [x]_Z$

(follows from definition)

Since for any equivalence class, $p^{-1}([x]_Z) \cap F$ in one point, take y to be that point. Then $p(y) = [y]_Z = [x]_Z$

1-1 We need to show that if $p|_F(x) = p|_F(y)$ then $x = y$. (ie $x, y \in F$)

Since $p(x) = p(y) = [x]_Z = [y]_Z$

Since F intersects each equivalence class exactly once, we have that $x = y$

:)