Rotation Dynamics

What do we gain from the perspective of the circle as a set of equivalence classes?

Recall! The circle can be thought of as the set of equivalence classes

\[ [x] = \{ \ldots, -4\pi + x, -2\pi + x, x, x + 2\pi, \ldots \} \]

"such that" \( x \) the integers

\[ \{ k2\pi + x : k \in \mathbb{Z} \} \]

need to tell what the conditions on \( k \) are

in words:

"the set of all numbers \( 2\pi k + x \), where \( k \) is an integer"
yesterday: the geometry of equiv. classes
today: the algebra of equivalence classes.

Idea: Develop addition on equivalence classes

Define: \( A + B := \{ x + y : x \in A, y \in B \} \)

example:

\[
A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}
\]

\[
A + B = \{ 1 + 3, 1 + 4, 2 + 3, 2 + 4 \}
\]

\[
= \{ 4, 5, 5, 6 \} \quad \text{< duplicates listed once}
\]

Theorem: For any equiv. classes \([x]_E, [y]_E\),

\([x]_E + [y]_E = [x + y]_E\).

Proof: We’ll show that \([x]_E + [y]_E \leq [x + y]_E\) and

that \([x + y]_E \leq [x]_E + [y]_E\).
First: choose some $z \in [x]+[y]$.  

By definition of set addition,

$$Z = (x + k \cdot 2\pi) + (y + l \cdot 2\pi)$$

for some $k, l \in \mathbb{Z}$.  

$$= (x + y) + 2\pi(k + l)$$

Since $k + l$ is an integer, $z \in [x+y]$.  

Now, we show that $[x+y] \subset [x]+[y]$.  

Let $w \in [x+y]$.  

So by definition,

$$w = x + y + 2\pi m$$

Can split $m$ into the sum of any two integers whose sum is $m$.  Simplest choice is $0$ and $m$,

$$w = (x) + (y + 2\pi m)$$

Since $x \in [x]$ and $y + 2\pi m \in [y]$, we have
that \( w \in [x] + [y] \).

\[ \rightarrow \square \]

means we've proved the thing we wanted to prove.

Observations

**identity property**

\((a) \ [0] + [x] = [x] \)

**associativity property**

\((b) \ ([x] + [y]) + [z] = [x] + ([y] + [x]) \)

**existence of inverses**

\((c) \ [-x] + [x] = [0] \)

these properties give that \( \mathbb{R}/2\pi \mathbb{Z} \)

a group with the operation of set addition

**bonus property**

\((d) \ [x] + [y] = [y] + [x] \)
Rotation Dynamics

Define \( R_\theta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z} \)

\[ R_\theta([x]) = [x+\theta] = [x] + [\theta] \]

This is exactly the circle rotation by angle \( \theta \).

**Goal:** "Theorem"  

\[ \text{Thm} \] If \( \theta = \frac{p}{q} \cdot 2\pi \), every point on the circle has period \( q \).

\[ \text{Thm} \] If there exists a periodic point \( p \in \mathbb{Z} \) of period \( q \) for the circle rotation by \( \theta \), then \( \theta = 2\pi \frac{p}{q} \) for some \( p \in \mathbb{Z} \).
Each orbit has 3 points and are disjoint.

A formula for $R_{\theta}^k([x])$

$R_{\theta}^k([x]) = [x + k\theta]$ for all $k \in \mathbb{Z}$, $k \geq 1$.

**Proof (by induction)**

1. Write a proof for the base case.
2. Then prove if I know previous one, I can deduce the next one.

Works like dominos.
Proof of base case

Check that \( R_\theta([x]) = [x + 1\theta] \)

true from the definition

Proof of inductive step) Assume that it holds for \( K \) and prove that it holds for \( K+1 \).

Assume it holds for \( R_\theta^K([x]) = [x + K\theta] \)

by def of iterates

Then \( R_\theta^{K+1}([x]) = R_\theta(R_\theta^K([x])) \)

by inductive assump

\[
= R_\theta([x + K\theta])
\]

by def of \( R_\theta \)

\[
= [x + K\theta] + [\theta]
\]

property of set addition

\[
= [x + K\theta + \theta]
\]

factoring

\[
= [x + \theta(K+1)]
\]

\( \square \)