preREU - Last week projects

Problem 1 (Expanding maps)**.** This problem will use the following as a black box:

Theorem. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a continuous function. Then there exists a unique continuous function $F : \mathbb{R} \to \mathbb{R}$ and integer $d \in \mathbb{Z}$ such that $0 \le F(0) < 1$, $f([(x)]) = [F(x)]$ (ie, $f \circ p = p \circ F$) and $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$. The number d is called the *degree* of f, and denoted $d = \deg(f)$ and F is called the *lift* of f.

The goal of this project is to build an invertible map $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that

$$h \circ f = L_d \circ h \qquad (\clubsuit)$$

when f has degree d and satisfies a certain condition.

A) Show that if $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is invertible, then $\deg(h) = \pm 1$.

B) A map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is called *expading* if there exists $\lambda > 1$ such that $\left|F'(x)\right| \ge \lambda$ for all $x \in \mathbb{R}$. Show that if f is expanding, then $|\deg(f)| > 1$.

C) Show that if $H : \mathbb{R} \to \mathbb{R}$ is a function such that

$$H(x) = \frac{1}{d} H(F(x)) \qquad \text{and} \qquad H(x+1) = H(x) + 1 \text{ for all } x \in \mathbb{R},$$

then $h([x]) := [H(x)]$ is well-defined and satisfies equation ($\clubsuit$).

D) Define $\mathcal{F}(H)(x) = \frac{1}{d} H(F(x))$ as a dynamical system on continuous functions. Show that $\mathcal{F}$ preserves the set of functions H such that $H(x+1) = H(x) + 1$, and that there exists $\mu < 1$ such that for every pair of functions H_1 and H_2 and $x \in \mathbb{R}$,

$$|\mathcal{F}(H_1)(x) - \mathcal{F}(H_2)(x)| \le \mu \, |H_1(x) - H_2(x)| \, .$$

E) Look up the *contraction mapping principle*, and use it to show that the conjugating map h exists and is unique.

Problem 2 (Linear flows)**.** The *two-dimensional flat torus* is the set of equivalence classes of $\mathbb{R}^2/\mathbb{Z}^2$. That is, we define the equivalence class

$$\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \left\{ \begin{pmatrix} x+m \\ y+n \end{pmatrix} : m, n \in \mathbb{Z} \right\}.$$

If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ is a nonzero vector, the *flow* generated by v is the dynamical system

$$\varphi_t\left(\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]\right) = \left[\begin{pmatrix} x + tv_1 \\ y + tv_2 \end{pmatrix}\right]$$

A) Find a fundamental domain for $\mathbb{R}^2/\mathbb{Z}^2$.

B) Show that

$$\varphi_t \circ \varphi_s = \varphi_{t+s} \qquad (\heartsuit)$$

for all $t, s \in \mathbb{R}$. Equation ($\heartsuit$) is called the *flow equation.*

C) Show that if φ_t is the flow generated by $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and ψ_t is the flow generated by $\begin{pmatrix} v_2 \\ v_1 \end{pmatrix}$, then φ_t is conjugated to ψ_t.

D) Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ have $v_2 > 0$ and $Y = \left\{ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}\right] : x \in \mathbb{R} \right\}$. Find the first return map and first return time to Y.

E) Describe the orbits of φ_t based on the vector v. Can you make sense of these questions: When are the orbits periodic? When are the orbits dense?

Problem 3 (Convergents and continued fractions)**.** Let $\alpha \in (0,1)$ have continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

and p_k/q_k be the convergent

$$\frac{p_k}{q_k} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}}$$

A) Define the vector $v_k = \begin{pmatrix} p_k \\ q_k \end{pmatrix}$ and the matrix $B(a) = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$. Show that

$$(\spadesuit) \qquad v_k = B(a_1)B(a_2)\dots B(a_k)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

B) Show that if α has periodic continued fraction expansion, with repeating digits $a_1, \dots, a_k$, then $v_0 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ is an eigenvalue of

$$A_k = B(a_1)B(a_2)\dots B(a_k).$$

C) Use the formula $(\spadesuit)$ to find a recursive definition of p_k and q_k, and prove that your formula holds by induction. [*Hint*: You will need a 2-step induction]

D) Show by induction that the trace of A_k is an integer than 1 and that the determinant of A_k is equal to ± 1.

E) Show that any matrix of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 such that $0 \le a \le c$, $a \le b$, and $0 \le b \le d$, then A is a product of matrices of the form $B(a_1)\dots B(a_k)$.

Problem 4 (Local dynamics and Morse-Smale Systems)**.** Let $(a,b) \subset \mathbb{R}$ be an interval, $x \in (a,b)$ and $f : (a,b) \to \mathbb{R}$ be a continuously differentiable function such that $f(x) = x$.

A) Show that if $f'(x) < 1$, then there exists $\varepsilon > 0$ such that if $|y - x| < \varepsilon$, then $\lim_{n\to\infty} f^n(y) = x$.

B) Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a dynamical system, and assume that some interval $[a,b]$ is invariant under f. Show that f has a fixed point in $[a,b]$

C) A dynamical system on $\mathbb{R}/\mathbb{Z}$ is called *Morse-Smale* if it is invertible, there exists finitely many fixed points, and at each fixed point p, $f'(p) > 0$ and $f'(p) \neq 1$. Find examples of Morse-Smale dynamical systems.

D) Show that it is impossible for a Morse-Smale system to have only one fixed point. [*Hint*: Use **B)** to find another fixed point]

E) Show that if f and g are Morse-Smale dynamical systems on $\mathbb{R}/\mathbb{Z}$ with exactly two fixed points, then they are conjugated. (This type of system is said to have *north-south pole dynamics*)