Theorem Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an irrational rotation. **Day 15**

Then the first return map on $[0,\alpha)$ is conjugated to $R_{-\gamma}$, where $\gamma = G(\alpha)$ and

$$G(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor$$

G is called the Gauss map.
It can be defined
$$G : [0,1)\setminus\mathbb{Q} \to [0,1)\setminus\mathbb{Q}$$
$$y \longmapsto \frac{1}{y} - \left\lfloor \frac{1}{y} \right\rfloor$$

$R_{-\gamma}$ is a rotation on $[0,1)$

Conjugated?

$$\begin{tikzcd} {[0,1)} \arrow[r, "R_{-\gamma}"] \arrow[d, "H"'] & {[0,1)} \arrow[d, "H"] \\ {[0,\alpha)} \arrow[r, "F"'] & {[0,\alpha)} \end{tikzcd}$$

Proof of the theorem

Define $n = \left\lfloor \frac{1}{\alpha} \right\rfloor$. Since α is irrational, so is $\frac{1}{\alpha}$, and $n < \frac{1}{\alpha} < n+1$.

Hence, $n < \frac{1}{\alpha} \implies n\alpha < 1$.

Let $\beta = 1 - n\alpha > 0$.

Observe also $(n+1)\alpha > 1$

Claim: If F is the first return map $F : [0,\alpha) \to [0,\alpha)$, then

$$F(x) = \begin{cases} R_\alpha^{n+1}(x), & 0 \le x < \beta, \\ R_\alpha^{n}(x), & \beta \le x < \alpha. \end{cases}$$

Proof of claim: • If $\beta \le x < \alpha$, then

$$R_\alpha^n([x]) = [x+n\alpha]$$

Since $x \geq \beta$, then $x + n\alpha \geq \beta + n\alpha$

but $\beta = 1 - n\alpha > 0$

So $x + n\alpha \geq \beta + n\alpha = (1-n\alpha) + n\alpha = 1$

then $x + n\alpha - 1 \geq 0$

Since $x < \alpha$, then $x + n\alpha < \alpha + n\alpha$

$< \alpha + 1$

So $x + n\alpha - 1 < \alpha$

Conclusion: $0 < x + n\alpha - 1 < \alpha$

& $R^n([x]) = [x + n\alpha - 1] = [x + n\alpha] \in [0, \alpha)$.

• If $0 \le x < \beta$, then

$$R_\alpha^{n+1}([x]) = [x + (n+1)\alpha]$$

Since $x \geq 0$, then $x + (n+1)\alpha \geq 0 + (n+1)\alpha \ge 0 + 1$

Since $x < \beta$, then $x + (n+1)\alpha < \beta + (n+1)\alpha$

but $\beta = 1 - n\alpha > 0$

So $x + (n+1)\alpha < (1 - n\alpha) + (n+1)\alpha$

$< 1 + \alpha$.

Therefore

$1 \le x + (n+1)\alpha < 1 + \alpha$

In conclusion,

$0 \le x + (n+1)\alpha - 1 < \alpha$

& $R^{n+1}([x]) = [x + (n+1)\alpha - 1]$

$= [x + (n+1)\alpha] \in [0, \alpha)$

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(Continuing with the proof of the theorem)

Define $H : [0,1) \to [0,1)$ by $H(x) = \alpha x$

Note that:

$$R_{-\gamma}(x) = \begin{cases} x - \gamma + 1, & 0 \leq x < \gamma. \\ x - \gamma, & \gamma \leq x < 1. \end{cases}$$

$$\begin{tikzcd} {[0,1)} \arrow[r, "R_{-\gamma}"] \arrow[d, "H"'] & {[0,1)} \arrow[d, "H"] \\ {[0,\alpha)} \arrow[r, "F"'] & {[0,\alpha)} \end{tikzcd}$$

It suffices to prove that $H(R_{-\gamma}(x)) = F(H(x))$ for all x

Proof of this:

Case 1 For $0 \leq x < \gamma$, we want to prove
$H(R_{-\gamma}(x)) = F(H(x))$

Computation:

$$\begin{aligned} LHS &= H(x - \gamma + 1) = \\ &= \alpha(x - \gamma + 1) \\ &= \alpha x - \alpha\gamma + \alpha \\ &= \alpha x - \alpha\left(\frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor\right) + \alpha \\ &= \alpha x - 1 + \left\lfloor \frac{1}{\alpha} \right\rfloor \alpha + \alpha \\ &= \alpha x - 1 + n\alpha + \alpha \\ &= \alpha x - 1 + (n+1)\alpha = \alpha x + (n+1)\alpha - 1 \end{aligned}$$

$\text{RHS} = F(\alpha x)$, but $0 \leq \alpha x < \alpha \gamma = \alpha\left(\frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor\right)$

So $\alpha x < 1 - n\alpha = \beta$

then $F(\alpha x) = \alpha x + (n+1)\alpha - 1$

Since LHS = RHS we are done with the case 1.

Case 2. **Exercise** For $\gamma \leq x < 1$, we want to prove $H(R_{-\gamma}(x)) = F(H(x))$

Remark: Observe that if $\beta = 1 - n\alpha$ solve for α?

$$n\alpha + \beta = 1$$

$$\alpha\left(n + \frac{\beta}{\alpha}\right) = 1$$

$$\alpha = \frac{1}{n + \frac{\beta}{\alpha}}, \quad \frac{\beta}{\alpha} = \frac{1}{\alpha} - n = G(\alpha)$$

$$\alpha = \frac{1}{n + G(\alpha)}$$

Definition $n_i = \left\lfloor \frac{1}{G^{i+1}(\alpha)} \right\rfloor$

Exercise Prove by induction, if $\alpha \in (0,1) \setminus \mathbb{Q}$ i.e. $\alpha \in (0,1)$ & $\alpha \notin \mathbb{Q}$.

Then
$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots + \cfrac{1}{n_k + G^k(\alpha)}}}}}$$

Definition Iterational number

$$\frac{p_k}{q_k} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots \cfrac{1}{n_{k-1} + \cfrac{1}{n_k}}}}}$$

are called the convergents of α.

Theorem $\frac{p_k}{q_k} \longrightarrow \alpha$ as $k \to \infty$

Exercise: Review this

we won't prove this.

Theorem Let $\alpha \in (0,1) \setminus \mathbb{Q}$, and $\frac{p_k}{q_k}$ be a convergent of α. For any fraction $\frac{m}{n}$ such that $0 < n < q_k$, then

$$|q_k \alpha - p_k| < |n\alpha - m|$$

Moral: q_k are the times for rotations of 0 by α, when we get closer than we have ever been.

Theorem: (Dirichlet Approximation Theorem)
Let $\alpha \in [0,1) \setminus \mathbb{Q}$. There exists infinitely many fractions $\frac{p}{q}$ such that

$$|q\alpha - p| < \frac{1}{q} \quad \left(\iff \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}\right)$$

Proof: Fix an integer n, and divide $[0,1)$ into n equal pieces

$$[0,1) = [0,\tfrac{1}{n}) \cup [\tfrac{1}{n},\tfrac{2}{n}) \cup \dots \cup [\tfrac{n-1}{n},1)$$

Let $A = \{[0], R([0]), \dots, R^n([0])\}$

There are $n+1$ elements of A.

Since there are n intervals, there exists two number $0 \leq k < l \leq n$ such that $R^k([0])$ and $R^l([0])$ both belong to the same interval.

Hence, $|k\alpha - l\alpha - p| < \frac{1}{n}$ for some p

if $q = k - l$, then

$$|q\alpha - p| < \frac{1}{n}$$

and $\quad 1 \leq q \leq n$

Thus $\quad |q\alpha - p| < \frac{1}{n} \leq \frac{1}{q}$

Exercise: Conclude the Dirichlet approx Theorem