From now on, with $[x]$ we mean $[x]_2$. i.e. $[x] \in \mathbb{R}/\mathbb{Z}$.

we proved that $p^{-1}(\Theta_\pm([0]))$ is a subgroup of $\mathbb{R}$.

**Goal:** Theorem Every subgroup of $\mathbb{R}$ is either

1) $c\mathbb{Z}$ for some $c \geq 0$
2) dense.

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**More Properties of Subgroups of $\mathbb{IR}$**

**Lemma:** If $G \subset \mathbb{R}$ is a subgroup, and $c \in G$ then $c\mathbb{Z} \subset G$

**Proof:** Step 1: We will show that for all $n \geq 1$, $cn \in G$.

Proof of Step 1: By induction...

Base: Since $c \in G$, $1 \in G$, and $C = C \cdot 1 \in G$

Inductive: Assume that $cn \in G$

Then $c(n+1) = cn + c \cdot 1 = cn + c$

Since $cn \in G$ & $c \in G$ by closure property of $G$ (it is a group) we have $c(n+1) = cn + c \in G$
Step 2: For all \( n \leq \), \( c_n \in G \)
Since 6 is a group if \( c_n \in G \) for \( n \leq 1 \)
So it inverse. Therefore \(-c_n \in G\) and \(-c_n = c(-n) \in G\).

Step 3: when \( n = 0 \), \( c_n = c_0 = 0 \). So
\( 0 = c_0 \in G \), since the identity
is an element of every subgroup.

**Exercise** If \( c \in \mathbb{R} \), \( c \mathbb{Z} = (-c) \mathbb{Z} \)

**Lemma**: If \( c > 0 \) and \( n \in \mathbb{Z} \), there exists
some \( y \in \mathbb{Z} \) such that \( |x - y| < c \)

Proof of lemma: Let \( n \) be the largest integer
such that \( nc \leq x \). Let \( y = nc \).
Since \( n \) was the largest integer such that
\( nc \leq x \), it follows that
\( x \leq (n+1)c \).
Therefore,

\[ |x - y| = |x - nc| = x - nc \leq (n+1)c - nc = c \]

**Exercise** Show that if \( n \in \mathbb{Z} \) is the smallest integer such that \( cn > x \), then \( |x - cn| < c \)

**Theorem** If \( C \) is not dense, then there exists \( c > 0 \), such that \( (0, c) \cap C = \emptyset \).

**Proof** We will prove this by contrapositive! i.e., we will assume that if for every interval \( (0, c) \), \( C \cap (0, c) \neq \emptyset \), then conclude that \( C \) is dense.

Let \( (a, b) \subset \mathbb{R} \) an arbitrary interval.

We will show \( (a, b) \cap C \neq \emptyset \).

Let \( x = \frac{a+b}{2} \) the midpoint of \( (a, b) \)

\[ \varepsilon = \frac{b-a}{2} \] the distance of the midpoint \( x \) to \( a \) and \( b \).

Show \( (a, b) \cap C \neq \emptyset \) is equivalent to show \( (x - \varepsilon, x + \varepsilon) \cap C \neq \emptyset \)

By assumption, there exists \( \varepsilon \in C \cap (0, \varepsilon) \neq \emptyset \)
By the first lemma, \( S \cap \mathbb{R} < \varepsilon \).
By today's second lemma, there exists \( y \in S \cap \mathbb{R} \) s.t. \(|x - y| < \varepsilon\)

Finally, \( y \in \mathbb{G} \) (because \( S \cap \mathbb{R} < \varepsilon \)) because \( \mathbb{G} \subseteq \mathbb{G} \cap (0, \varepsilon) \) then \( \varepsilon < \varepsilon \).
we have that

\[ |x - y| < \varepsilon < \varepsilon. \]

That is \( y \in (x - \varepsilon, x + \varepsilon) \)

but \( y \in \mathbb{G} \) as well, so:

\[ y \in \mathbb{G} \cap (x - \varepsilon, x + \varepsilon) \neq \emptyset. \]

Exercise 60: Over the proof of the theorem again.