Recall: A subset \( A \subset \mathbb{R} \) is dense if for every interval \((a, b)\), then \((a, b) \cap A \neq \emptyset\).

Back to circle rotations...

What does it mean for an orbit (see) to be dense on the circle?

Still want no gaps...

Define \( p: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \), \( p(x) = [x]_\mathbb{Z} \)

Exercise: Show that "function" \( f([x]_\mathbb{Z}) = x \)

is not well defined (not possible)
$P$ is called the projection from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$.

If $A \subseteq \mathbb{R}/\mathbb{Z}$, let

\[ p^{-1}(A) := \{ x \in \mathbb{R} : p(x) \in A \} \]

The gaps in $A$ are the same as the gaps in $p^{-1}(A)$.

But not vice versa.

**Exercise** Show that $\mathbb{Q}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$ where $\mathbb{Q}/\mathbb{Z} = \{ [\frac{q}{n}]_{\mathbb{Z}} : \frac{q}{n} \in \mathbb{Q} \}$

**Exercise** Show that any finite subset of $\mathbb{R}/\mathbb{Z}$ is not dense in $\mathbb{R}/\mathbb{Z}$. 
Recall of definition of group

A group is a set $G$ with an operation $\ast$ such that $\forall g, h \in G$ then $g \ast h \in G$ and:

1) $\exists e \in G$ s.t. $\forall g \in G$, $e \ast g = g = g \ast e$ (identity)
2) $\forall g, h, k \in G$ s.t. $(g \ast h) \ast k = g \ast (h \ast k)$ (associativity)
3) $\forall g \in G$, $g^{-1} \in G$ s.t. $g \ast g^{-1} = g^{-1} \ast g = e$ (inverse)

Example: Odd integers are not a subgroup, of odds

Example: $\mathbb{Q} \subseteq \mathbb{R}$ is a subgroup

Example: $\mathbb{Z}/2 \mathbb{Z}$ is not a subgroup, since there are no inverses

Theorem: Let $\theta : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the dynamics of rotation by $\alpha$. Then $\rho'(\Theta_{\pm}(\mathbb{Z}/2))$ is a subgroup of $\mathbb{R}$, where $\Theta_{\pm}(\mathbb{Z}/2) = \{ n\alpha(\mathbb{Z}/2) : n \in \mathbb{Z} \}$.

Proof: Denote $H = \rho'(\Theta_{\pm}(\mathbb{Z}/2))$

Step 1: We claim that $H = \{ k + l\alpha : k, l \in \mathbb{Z} \}$. 

proof of claim: Suppose \( x \in H \) i.e. \( x = p'(\theta_2[0]_2) \) 

by definition of set pre-image 

since \( x \in p^{-1}(\theta_2[0]_2) \) 

\[ p(x) = [x]_2 \in \theta_2[0]_2 \] 

then there exists \( l \in \mathbb{Z} \) s.t. 
\[ [x]_2 = R^l_\alpha ([0]_2) \] 

Now 
\[ R^l_\alpha ([0]_2) = [0 + lx]_2 = [lx]_2 \] 

so 
\[ [x]_2 = [lx]_2 \] 

This means that there exists \( k \in \mathbb{Z} \) s.t. 
\[ x = k + lx. \]

This proves \( x = k + lx \in H = \{ k + lx : k, l \in \mathbb{Z} \} \)

i.e. \( p'(\theta_2[0]_2) = H \subset \{ k + lx : k, l \in \mathbb{Z} \} \)

**Exercise** prove that \( \{ k + lx : k, l \in \mathbb{Z} \} \subset p'(\theta_2[0]_2) \)

**Step 2** Now we prove that \( H \) is a subgroup.

check: \( (k + l \alpha) + (m + n \alpha) = (m + k) + (l + n) \alpha \in H \)
check: \( 0 \in H ? \) yes \( 0 = 0 + 0 \cdot \alpha \in H \)
check: let \( x \in H \), is \(-x\) in \( H \)? 

yes, if \( x = k + l \alpha \) for \( k, l \in \mathbb{Z} \) 
\[ -x = -(k + l \alpha) = -k + (-l) \alpha \in H \]