Thinking Dynamically

A (discrete-time) dynamical system is a function from a set $X$ to itself.

$$f : X \rightarrow X$$

Same Space

(Things that get plugged in)

(Things that come out)

**Example:** $X = \text{all real numbers}$ aka the number line aka $\mathbb{R}$

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ is } f(x) = \frac{1}{2}x$$

**Orbits:** The evolution of the system, for the point $x_0 = 6$ is

$$x_0 = 6 \Rightarrow x_1 = f(x_0) = \frac{1}{2}(6) = 3 \Rightarrow x_2 = f(x_1) = \frac{1}{2}(3) = \frac{3}{2} \Rightarrow x_3 = f(x_2) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$$

These orbits are related to the geometric sequence

**Notation:** $f^K(x) = \text{ } K^{\text{th}} \text{-step in the orbit of } x$.

**Warning:** $f^K(x) \neq (f(x))^K$

- The (forward) orbit of a point $x$ is the set $\Theta(x) = \{x, f(x), f^2(x), ...\}$
Example: \( X = \) a finite set \( \{a, b, c\} \)

\( f: X \to X \) is defined by the arrows in the following graph:

- \( f(a) = b \)
- \( f(b) = c \)
- \( f(c) = a \)

Orbits:
- \( a, f(a) = b, f^2(a) = f(f(a)) = f(b) = c, f^3(a) = a \)
- \( b, f(b) = c, f^2(c) = a, f^3(b) = b \)
- \( c, f(c) = a, f^2(c) = b, f^3(c) = c \)
in this case \( f^3 = \text{id} \) — identity transformation

Definition: A point \( p \) is called periodic if \( f^n(p) = p \) for some positive integer \( n \). The smallest such integer is called the period.

Example: For \( f(x) = \frac{1}{2}x \) has only one periodic point, \( x = 0 \).

Example: \( x = 3a, b, c \)

\[ g : x \rightarrow x \]

\( a \) and \( b \) and \( c \) are periodic with period 2.

\( a \) is preperiodic:

\[ f^2(a) = f^4(a) \]

Definition: A point \( x \) is pre-periodic if there are two integers \( m \neq n \) with

\[ f^m(x) = f^n(x) \]

Example: \( x = \text{unit circle} = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = 1\} \)

pick an angle \( \theta \) measured in radians.
$f : X \rightarrow X$ is the counter-clockwise rotation of $x$ at angle $\theta$.

Also called circle rotation with angle $\theta$.

If $\theta = \pi$, then

Example: The space of finite subsets of rational numbers

$p \in X$ can be $p = \{ \frac{1}{3}, \frac{2}{5}, \frac{4}{7} \}$

$f$ is defined by (for example):

$f \left( \frac{1}{3}, \frac{2}{5}, \frac{4}{7} \right) = \left\{ \frac{1}{3}, \frac{1+2}{3+5}, \frac{2}{5}, \frac{2+4}{5+7}, \frac{4}{7} \right\}$

$\uparrow$

Input: is a finite set in increasing order

Output: the initial set and the medians of two consecutive initial elements.
Some interesting thoughts:

- Start with any pair \(\{a, b\}\). Do you eventually see all rational numbers between \(a\) and \(b\)?
- What happens to the gaps between successive elements?
- What is the denominator size? How quickly do they grow?