# Thinking Dynamically

A (discrete-time) dynamical system is a function from a set $X$ to itself

$$f : X \to X$$

function

space (things that get plugged in)

same space (thing that come out)

**Example:** $X$ = all real numbers aka the number line aka $\mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ is $f(x) = \frac{1}{2}x$

**Orbits:** the evolution of the system, for the point $x_0 = 6$ is

$x_0 = 6$, $x_1 = f(x_0) = \frac{1}{2}(6) = 3$, $x_2 = f(x_1) = \frac{1}{2}(3) = \frac{3}{2}$

$x_4 = f(x_3) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$

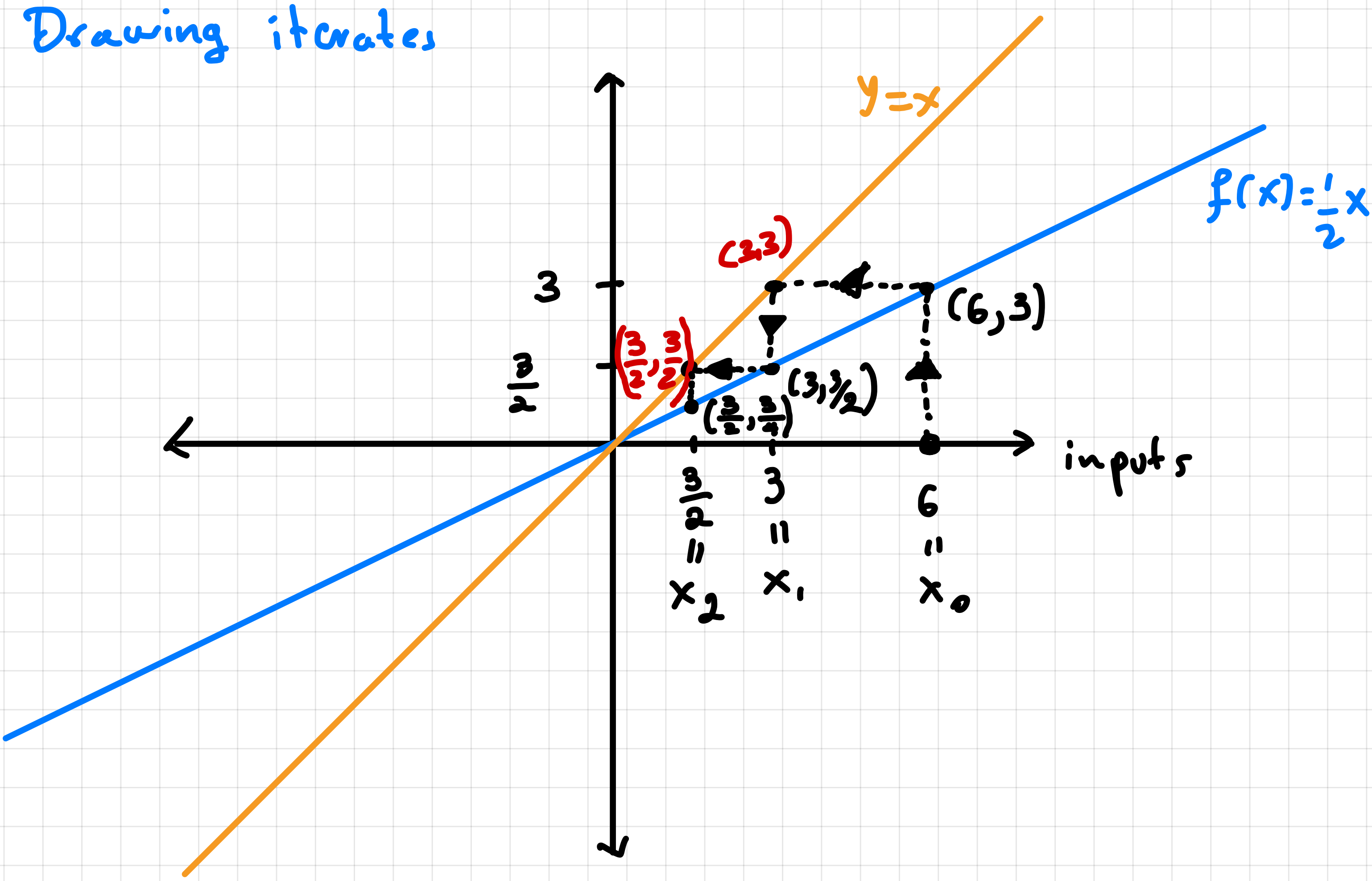
These orbits are related to the geometric sequence

**Notation:** • $f^k(x)$ = $k^{th}$-step in the orbit of $x$.

Warning: $f^k(x) \neq (f(x))^k$

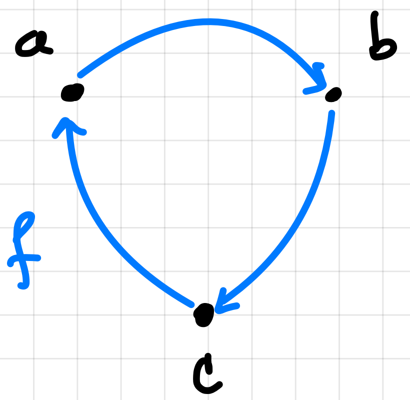
• The (forward) orbit of a point $x$ is the set $\mathcal{O}(x) = \{x, f(x), f^2(x), \dots\}$

# Drawing iterates




**Example**: $X$ = a finite set $\{a, b, c\}$

$f : X \to X$ is defined by the arrows in the following grap




$f(a) = b$

$f(b) = c$

$f(c) = a$

## Orbits

$a$, $f(a) = b$, $f^2(a) = f(f(a)) = f(b) = c$, $f^3(a) = a$

$b$, $f(b) = c$, $f^2(c) = a$, $f^3(b) = b$

$c$, $f(c) = a$, $f^2(c) = b$, $f^3(c) = c$

in this case $f^3 = id$ ← identity transformation

**Definition** A point $p$ is called *periodic* if $f^n(p) = p$ for some positive integer $n$

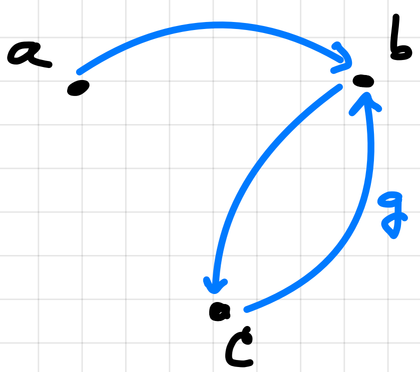
The smallest such integer is called the period.

**Example** for $f(x) = \frac{1}{2}x$ has only one periodic point, $x = 0$.

**Example**

$X = \{a, b, c\}$

$g : X \to X$




b & c are periodic with period 2.

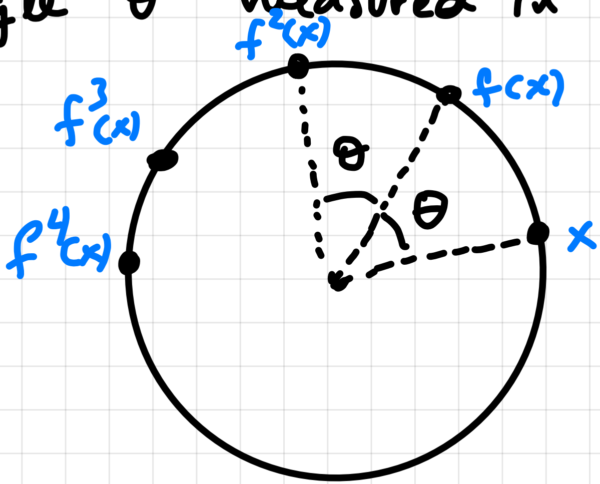
a is preperiodic: $f^2(a) = f^4(a)$

**Definition** a point $x$ is *pre-periodic* if there are two integers $m \neq n$ with

$$f^m(x) = f^n(x)$$

**Example** $X$ = unit circle $= \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$

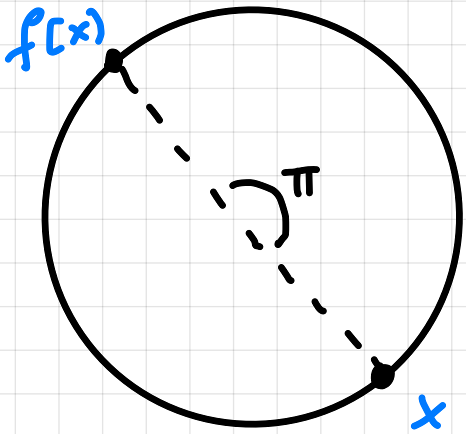
Pick an angle $\theta$ measured in radians



$f : X \to X$ is the counter-clockwise rotation of $x$ at angle $\theta$.

Also called circle rotation with angle $\theta$.

If $\theta = \pi$, then




Example $X$ = The space of finite subsets of rational numbers

$p \in X$ then $p$ can be $p = \{ \frac{1}{3}, \frac{2}{5}, \frac{4}{7} \}$

$f$ is defined by (for example):

$$f\left(\left\{ \frac{1}{3}, \frac{2}{5}, \frac{4}{7} \right\}\right) = \left\{ \frac{1}{3}, \frac{1+2}{3+5}, \frac{2}{5}, \frac{2+4}{5+7}, \frac{4}{7} \right\}$$

$$= \left\{ \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7} \right\}$$

input: is a finite set in increasing order

output: the initial set and the medians of two consecutive initial elements.

Some interesting thoughts:

- Start with any pair $\{a, b\}$. Do you eventually see all rational numbers between a and b?
- what happens to the gaps between succesive elements?
- what is the denominator size? How quickly do they grow?