RIGIDITY OF HIGHER RANK GROUP ACTIONS IN CONTINUOUS TIME

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These notes are divided into 3 parts, corresponding to the 3 lectures for the minicourse given at the Brin Summer School in June of 2023. The notes cover progress made by the author, many of which are in collaboration with Danijela Damjanovic, Ralf Spatzier and Disheng Xu. This writeup is intended as a quick introduction to tools used in smooth rigidity of hyperbolic abelian group actions, and makes no effort to provide a history or context. While references and historical context are not provided, feel free to email me for some! (vinhage@math.utah.edu)

The setting of these notes is as follows, and the notation will be kept throughout except where otherwise noted. All terms will be eventually defined.

G	A (often semisimple) matrix group
Γ	A cocompact, discrete subgroup of G
A	An abelian subgroup of G, (often the Cartan subgroup)
M	A compact subgroup of G commuting with A
$\alpha, \beta, \dots$	Roots of G with respect to A OR Lyapunov funtionals on A
W^α	Coarse Lyapunov foliation associated to the Lyapunov functional α
U^α	Groups parameterizing the leaves of W^α
X	A C^∞ manifold

1. Understanding the Setting and Models

1.1. **Rigidity Theorems.** A working definition of a rigidity theorem is a classification theorem for which the reader's reaction is "Wait, that can't be true..." Expereince will suggest that the situation should have many more possibilities than what's described. This is often reflected because the models are few, or much nicer than expected.

Here is a family of rigidity results and an associated conjecture which fits the bill.

Theorem 1.1 (Ghys,Kanai,Benoist-Foulon-Labourie,Feres,...)**.** *Let X be a C^∞ manifold such that the sectional curvature of X is always between -1 and -4. If the stable and unstable distributions of the geodesic flow are C^∞, then X is a locally symmetric space.*

Conjecture 1.2. *For any negatively curved manifold, if the stable/unstable distributions of the geodesic flow are of class C^∞, then X is locally symmetric.*

In the setting above, the "ideal" models are geodesic flows on locally symmetric spaces, and a seemingly innocuous assumption is imposed: the regularity of the stable and unstable manifolds.

These notes are about a rigidity program for abelian group actions. The "ideal" models here are affine transformations on bi-homogeneous spaces, and our innocuous assumption will be that the transformation is the restriction of a larger smooth group action. In particular, we have the following theorems

Theorem 1.3 (Spatzier-Vinhage)**.** *Let $\mathbb{R}^k \curvearrowright X$ be a C^∞ cone-transitive, totally Cartan action without rank one factors. Then up to finite cover, there is a C^∞ diffeomorphism $h : X \to H/\Gamma$ conjugating the action of A to a translation action on H/Γ.*

Theorem 1.4 (Damjanovic-Spatzier-Vinhage-Xu)**.** *Let G be a semisimple Lie group such that every simple factor has rank at least 2. Let $G \curvearrowright X$ be a C^∞, totally Anosov action of a semisimple Lie group with an ergodic invariant volume on X. Then up to finite cover, there is a C^∞ diffeomorphism $h : X \to K\backslash H/\Gamma$ conjugating the action of A to a translation action on $K\backslash H/\Gamma$.*

1.2. **Homogeneous spaces, metrics and hyperbolicity.** For the remainder of Section 1, we will focus on understanding the ideal models in this setting, and the assumptions which appear in Theorems 1.3 and 1.4. Let us begin with the obligatory first example, $SL(d,\mathbb{R})$. Readers with experience in Lie groups, Lie algebras and homogeneous spaces may skip this section. As we wish to study smooth ergodic theory for translation actions on $SL(d,\mathbb{R})/\Gamma$ for a cocompact subgroup Γ, we first wish to undersatnd how vectors expand and contract. Accordingly, we need a reference metric on $SL(d,\mathbb{R})/\Gamma$.

A first instinct may be to use the ambient metric, viewing $SL(d,\mathbb{R}) \subset \mathbb{R}^{d^2}$ as a (d^2-1)-dimensional submanifold, but since this is not invariant under the right-translation action by Γ, it doesn't descend as a well-defined metric on the quotient $SL(d,\mathbb{R})/\Gamma$. Instead, our strategy will be to choose any inner product $\langle\cdot,\cdot\rangle_e$ on the vector space T_eG, and the define the metric at g to be

$$\langle Y_1, Y_2\rangle_g := \langle dR_g(Y_1), dR_g(Y_2)\rangle_e,$$

where $R_g : SL(d,\mathbb{R}) \to SL(d,\mathbb{R})$ is right translation $R_g(h) = hg$ (we similarly define $L_g : SL(d,\mathbb{R}) \to SL(d,\mathbb{R})$ as $L_g(h) = gh$, the left translation).

Exercise 1.1. Show that the metric defined above is invariant under R_g for every $g \in G$, and that any such right-invariant metric is built this way.

Before computing (norms of) derivatives of left translations with respect to this metric, let's also understand what the tangent bundles look like. Here is a trick that works in computing the tangent spaces of Lie groups defined through a relation: let $\varphi : (-\varepsilon,\varepsilon) \to SL(d,\mathbb{R})$ be a curve contained in $SL(d,\mathbb{R})$ passing through e. Since $SL(d,\mathbb{R})$ is uniquely defined via $\det(A) = 1$, we have that $\det(\varphi(t)) \equiv 1$. Differentiating this at $t = 0$ via the chain rule, and noting that $D(\det) = \mathrm{Tr}$ at $e \in SL(d,\mathbb{R})$ (this is a linear algebra fact), we get that

$$\mathrm{Tr}(\varphi'(0)) = 0.$$

Hence, the tangent space at the identity is the set of traceless matrices. Finally, observe that since the group operation is matrix multiplication, the derivative of the translation action is just again matrix multiplication. Then, if $Y \in T_eSL(d,\mathbb{R})$ is a tracless matrix and $g \in SL(d,\mathbb{R})$

$$||dL_g(Y)||_g = \left|\left|dR_{g^{-1}} dL_g(Y)\right|\right|_e = \left|\left|gYg^{-1}\right|\right|.$$

This is the main feature of translation actions:

$$\boxed{\text{Local behavior of translation dynamics is determined by the conjugation action}}$$

Let's try to make this a bit more precise. Notice that $C_g : Y \mapsto gYg^{-1}$ is linear in Y, and for simplicity, let's assume T_eG admits a splitting into eigenspaces $T_eSL(d,\mathbb{R}) = \bigoplus_{i=1}^{\ell} E^{\lambda_i}(e)$, where $gYg^{-1} = e^{\lambda_i}Y$.

Exercise 1.2. Show that if $E^{\lambda_i}(h) := dR_h E^{\lambda_i}(e)$, then it defined a vector subbundle of $SL(d,\mathbb{R})$ which descends to $SL(d,\mathbb{R})/\Gamma$. Furthermore, show that if $Y \in E^{\lambda_i}$, then $||dL_{g^n}(Y)|| = \lambda^n ||Y||$.

1.3. **Weyl chamber flow on $SL(d,\mathbb{R})$.** We wish to study group actions, but the tools developed in the previous section allow us to do exactly that. Consider the subgroup of diagonal matrices in $SL(d,\mathbb{R})$, and note that we get a condition on the entries since the determinant must be 1:

$$A = \left\{ \begin{pmatrix} e^{t_1} & & & \\ & e^{t_2} & & \\ & & \ddots & \\ & & & e^{t_d} \end{pmatrix} : \sum t_i = 0 \right\}$$

Each $a \in A$ has its own conjugation action on G, but these conjugations all commute since A is abelian. Hence (assuming the conjugations are all diagonalizable), $T_e SL(d,\mathbb{R})$ must admit a splitting into eigenspaces.

Exercise 1.3. Show that the subspaces $E_{ij} \subset T_e SL(d,\mathbb{R})$, those matrices whose entries are 0 except for the $(i,j)^{\text{th}}$ entry, are the joint eigenspaces, and that the eigenvalue of the conjugation action of $a = \operatorname{diag}(e^{t_1},\dots,e^{t_d})$ is exactly $e^{t_i - t_j}$.

Definition 1.5. The functionals $\alpha_{ij}(t) := t_i - t_j$ appearing in exponents of Exercise 1.3 are called the *Lyapunov exponents* or *Lyapunov functions* for the action. The are always the logarithms of the eigenvalues of conjugation action of A on a given invariant subspace. We also denote the subspaces by $E^{\alpha_{ij}}$, which is more easily adaptable to future examples

We have established a very nice dynamical picture for different elements of the A-action. In particular, if $a \notin \ker \alpha_{ij}$ for every $i \neq j$ (ie, no two entries of a are the same), then TX (where $X = SL(d,\mathbb{R})$) admits an A-invariant splitting

$$TX = E^s_a \oplus T\mathcal{O} \oplus E^u_a,$$

where $E^s_a = \bigoplus_{\alpha_{ij}(a)<0} E^{\alpha_{ij}}$ and $E^u_a = \bigoplus_{\alpha_{ij}(a)>0} E^{\alpha_{ij}}$, and $T\mathcal{O}$ is the orbit foliation of the A-action. This is a partially hyperbolic splitting for a (in fact, it is normally hyperbolic for a with respect to the orbit foliation of A), but crucially for our analysis, depends on the element a you choose. In fact, we have the following nice feature:

For the diagonal action on $SL(d,\mathbb{R})/\Gamma$, the set of elements normally hyperbolic to the orbit foliation is the complement of the hyperplanes determined by $\ker \alpha_{ij}$. In the cones bounded by these kernels (called walls), the stable and unstable distributions do not change, but the hyperbolicity is weaker closer to the walls. When one passes over a wall, at least one joint eigenspace passes from positive to negative.

1.4. **A more exotic group: The case of $SO(m,n)$.** Let's move to another group and make some comparisons. Fix $n \geq m$. If $v, w \in \mathbb{R}^{m+n}$, and Id_k denotes the $k \times k$ identity matrix, consider the symmetric bilinear form

$$\sigma(v,w) = v^T Q w, \qquad Q = \begin{pmatrix} \mathbf{0} & \mathrm{Id}_m & \mathbf{0} \\ \mathrm{Id}_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{Id}_{n-m} \end{pmatrix}$$

We say that a matrix g preserves the form if $\sigma(gv,gw) = \sigma(v,w)$, and let the group of matrices preserving σ be denoted by $SO(m,n)$ (this is because σ is a symmetric bilinear form with signature (m,n)).

Exercise 1.4. Show that g preserves σ if and only if $g^T Q g = Q$, and that the tangent space to the identity at $SO(m,n)$ is the set of matrices with block form

$$\begin{pmatrix} A & B & D \\ C & -A^T & E \\ -E^T & -D^T & F \end{pmatrix} \qquad B, C, F \text{ antisymmetric.}$$

Here, A, B and C are $m \times m$, D and E are $m \times (n-m)$ and F is $(n-m) \times (n-m)$. Recall that a matrix g is antisymmetric if $g = -g^T$. [*Hint*: Use the trick we did to find $T_e SL(d, \mathbb{R})$]

Using Exercise 1.4, we can write $T_e SO(2,4)$ as the set of matrices

$$\left(\begin{array}{cc|cc|cc} t_1 & s_{12} & 0 & u & \sigma_1 & \sigma_2 \\ s_{21} & t_2 & -u & 0 & \tau_1 & \tau_2 \\ \hline 0 & v & -t_1 & -s_{21} & \hat{\sigma}_1 & \hat{\sigma}_2 \\ -v & 0 & -s_{12} & -t_2 & \hat{\tau}_1 & \hat{\tau}_2 \\ \hline -\hat{\sigma}_1 & -\hat{\tau}_1 & \sigma_1 & \tau_1 & 0 & \theta \\ -\hat{\sigma}_2 & -\hat{\tau}_2 & \sigma_1 & \tau_2 & -\theta & 0 \end{array}\right)$$

where all variables are real. In this case the *split Cartan subgroup*, or maximal connected abelian subgroup of matrices whose conjugation action on $T_e G$ is diagonalizable over $\mathbb{R}$, is $A = \left\{ \operatorname{diag}(e^{t_1}, e^{t_2}, e^{-t_1}, e^{-t_2}, 1, 1) : t_i \in \mathbb{R} \right\}$.

Exercise 1.5. Find the joint eigenspaces and Lyapunov functionals for the action of A. Are they all 1-dimensional? Show that the matrices with only θ non-zero are fixed under the conjugation action of A, and are tangent to a subgroup isomorphic to $SO(2) := SO(0,2)$.

Observe that the action of A on $SO(2,4)/\Gamma$ does not have the same type of partially hyperbolic splitting. Instead, the subgroup isomorphic to $SO(2)$ inside $SO(2,4)$ commutes with A so the derivative of translations under A along it is isometric! In this case, we need to build a *double* quotient space $X = SO(2) \backslash SO(2,4)/\Gamma$ whose elements are double cosets $SO(2)g\Gamma$. Because we have quotiented by the subgroup isomorphic to $SO(2)$, the tangent space is spanned by all variables other than θ, and the translation action on X given by

$$a \cdot SO(2)g\Gamma = SO(2)ag\Gamma$$

is well-defined, and admits a partially hyperbolic splitting transverse to the orbit direction.

2. Anosov $\mathbb{R}^k$-actions and associated structures

Let's first generalize the structures we observed in the homogeneous setting to that of smooth actions.

Definition 2.1. Let $\mathbb{R}^k \curvearrowright X$ be a C^∞, locally free action on a compact manifold X.

- We say that $a \in \mathbb{R}^k$ is *Anosov* or *an Anosov element* if there exists an a-invariant splitting $TX = E^s_a \oplus T\mathcal{O} \oplus E^u_a$, where $T\mathcal{O}$ is the tangent bundle to the orbit foliation of $\mathbb{R}^k$, E^s_a is uniformly contracted under a and E^u_a is uniformly contracted under a^{-1}. That is, there exists $C > 0$, $0 < \lambda < 1$ (both depending on a) such that

$$||da^n(v)|| \leq C\lambda^n \text{ for all unit vectors } v \in E^s_a, n \geq 0$$
$$\left|\left|da^{-n}(v)\right|\right| \leq C\lambda^{-n} \text{ for all unit vectors } v \in E^s_a, n \geq 0$$

- An action $\mathbb{R}^k \curvearrowright X$ is Anosov if it has an Anosov element
- An action $\mathbb{R}^k \curvearrowright X$ is *totally Anosov* if it has a dense set of Anosov elements.

Exercise 2.1. Prove that the set of Anosov elements is open $\mathbb{R}^k$, and that if a is Anosov so is ta for all $t \in \mathbb{R} \setminus \{0\}$.

From the these assumptions do a lot for us in terms of constructing manifolds. In particular, for each Anosov element a, we get Hölder foliations with C^∞ leaves, W^s_a and W^u_a. The leaves are characterized by the dynamics of a:

$$W^s_a(x) = \{y \in X : d(a^n \cdot x, a^n \cdot y) \to 0\} \qquad W^u_a(x) = \left\{y \in X : d(a^{-n} \cdot x, a^{-n} \cdot y) \to 0\right\}$$

Since we are in higher rank, we hope to get more. In particular, we hope to get distributions and foliations which generalize the Lyapunov distributions. In general, while it won't always be possible with a precise rate, we can try to refine as best we can using the qualitative behavior.

Definition 2.2. Let $a_1, a_2, \dots, a_\ell$ be Anosov elements of $\mathbb{R}^k$. The distribution $E^s_{a_1,a_2,\dots,a_\ell} := \bigcap_{i=1}^{\ell} E^s_{a_i}$ is called a *joint stable distribution*. A joint stable distribution is called a *coarse Lyapunov distribution* if it contains no proper joint stable distribution of positive dimension.

Exercise 2.2. Show that coarse Lyapunov distributions always exist and are $\mathbb{R}^k$-invariant. [*Hint*: First show that each distribution E^s_a is $\mathbb{R}^k$-invariant]

The philosophy behind coarse Lyapunov foliations is to intersect manifolds defined dynamically and without rates to build as fine a splitting of TX as possible, so that we may analyze the structures there. It's not immediately obvious that the foliation whose leaves are the intersection of leaves of stable manifolds is again a foliation, since two stable leaves may not be of complementary dimension. However, we have the following

Proposition 2.3. *If $\mathbb{R}^k \curvearrowright X$ is an Anosov action, and $E_1, \dots, E_n$ are all of the distinct coarse Lyapunov foliations, then $TX = T\mathcal{O} \oplus \bigoplus_{i=1}^n E_i$ is an $\mathbb{R}^k$-invariant splitting. Furthermore, each bundle E_i is Hölder continuous, and integrates to a foliation with C^∞ leaves.*

Exercise 2.3. Show that if $TX = F_1 \oplus \dots F_m$ is an $\mathbb{R}^k$-invariant splitting and a is an Anosov element, then each F_m admit partially hyperbolic splitting (the stable/central/unstable bundles may be trivial in F_i).

Sketch of Proof of Proposition 2.3. Begin with an Anosov element a_1, and split $TX = T\mathcal{O} \oplus E^s_{a_1} \oplus E^u_{a_1}$. Consider another Anosov element a_2. If every choice of Anosov a_2 has the same (or opposite) splitting, then this splitting is already the coarse Lyapunov splitting of TX. Otherwise, a_2 is partially hyperbolic in each distribution $E^s_{a_1}$ and $E^s_{a_2}$. Since the central distribution for each Anosov a is always $T\mathcal{O}$, the a_2 acts hyperbolically on each of them, and there must exist at least one which has a nontrivial splitting.

Assume it is $E^s_{a_1}$. Then consider the fibered transformation on the space of pairs

$$\left\{(x, y) : x \in X, y \in W^s_{a_1}(x)\right\}.$$

Here, the usual construction of stable and unstable manifolds *looking only at the fiber* will go through, yielding subfoliaitons and distributions of $E^s_{a_1}$. We have thus built subfoliations of $W^s_{a_1}$ be refining $E^s_{a_1}$.

One now iterates this procedure until we can't anymore. When we can't, we've exactly arrived at the coarse Lyapunov splitting! ☐

Exercise 2.4. Assume that $A \curvearrowright G/\Gamma$ is a translation action on a homogeneous space G/Γ by a subgroup $\mathbb{R}^k \cong A \subset G \subset GL(d,\mathbb{R})$, and T_eG splits into joint eigenspaces $T_eG = T_eA \oplus \bigoplus_{\alpha\in\Delta} E^\alpha$, where $\Delta \subset A^*$ is a finite set of functions $\alpha : A \to \mathbb{R}$ and

$$E^\alpha = \left\{ Y \in T_eG : aYa^{-1} = e^{\alpha(a)}Y \right\},$$

then the coarse Lyapunov distributions are $E^{[\alpha]} = \bigoplus_{c>0} E^{c\alpha}$, $\alpha \in [\Delta]$, where

$$[\Delta] = \{\alpha \in \Delta : c\alpha \notin \Delta \text{ for all } c > 1\}.$$

Verify this directly for the Anosov actions on $SL(d,\mathbb{R})/\Gamma$ and $SO(m,n)/\Gamma$ (ie, for each coarse Lyapunov distriubtion, find a finite collection of Anosov elements which expresses it as a common stable manifold).

2.1. **Anosov vs. Totally Anosov actions.** For a general Anosov action, we may worry that we may not have "enough" Anosov elements to have refined enough. We may also wonder what the structure of the Anosov elements is. Under the *totally* Anosov condition, we get a complete description.

Proposition 2.4. *If $\mathbb{R}^k \curvearrowright X$ is a totally Anosov action, then the set of non-Anosov elements is a finite union of hyperplanes. Each hyperplane corresponds to either one or two coarse Lyapunov distributions. Furthermore, the connected components of Anosov elements are exactly the sets on which the splitting $TX = E^s_a \oplus T\mathcal{O} \oplus E^u_a$ is locally constant.*

Exercise 2.5. Show that for every Anosov $a \in \mathbb{R}^k$, E^s_a and E^u_a are sums of coarse Lyapunov distributions.

Sketch of Proof. For each coarse Lyapunov distribution E, let H_E^- denote the set of a in $\mathbb{R}^k$ such that for all $v \in E$, $da^n(v) \le C\lambda^n$ for some $C > 0$ and $0 < \lambda < 1$ (allowed to depend on a) and all $n \ge 0$. Similarly, let H_E^+ denote the set of all $a \in \mathbb{R}^k$ which are contracting in backward time.

Note that set of Anosov elements must be contained in $H_E^+ \cup H_E^-$ by Exercise 2.5. Hence, since the action is totally Anosov H_E^+ and H_E^- are dense. Finally, observe that $H_E^\pm$ are cones: invariant under positive scalar multiples and positive linear combinations (Another exercise! Just use that the elements commute, and preserve the distribution E).

In particular, by the Hahn-Banach separation theorem, there is a hyperplane that separates H_E^+ and H_E^-. Since $H_E^\pm$ are open cones, and dense, it must be their complement. That is, each $H_E^\pm$ is a half space. □

Exercise 2.6. Finish the proof of Proposition 2.4 using the fact that the sets $H_E^\pm$ are complementary open half-spaces and Exercise 2.5.

Remark 2.5. It is natural to ask whether every Anosov action is totally Anosov. Homogeneous and product examples are all totally Anosov once they are Anosov. There are examples of Anosov actions which are not totally Anosov. As of writing these notes, they only come from time changes of product examples, and we do not have an example for $\mathbb{Z}^k$-actions (where totally Anosov means that the Anosov elements are projectively dense).

We end this lecture with a definition which is a useful condition

Definition 2.6. An Anosov action is called *Cartan* if every coarse Lyapunov distribution is one-dimensional. It is called *totally Cartan* if it is totally Anosov and Cartan.

Exercise 2.7. Show that the homogeneous diagonal action on $SL(d,\mathbb{R})/\Gamma$ is totally Cartan, but the action on the biquotient $SO(n-m)\backslash SO(m,n)/\Gamma$ is not. For a harder example, find an example of a semisimple group for which the coarse Lyapunov distributions have more than one eigenspace (ie, α and $c\alpha$ are both Lyapunov functionals for some $c > 1$).

2.2. **The homogeneous case.** The case of homogeneous flows has particularly nice features for their coarse Lyapunov foliations. We first review a basic definition of Lie theory and a lemma immediately showing its usefulness. If $G \subset GL(d,\mathbb{R})$, and $Y_1, Y_2 \in T_eG$, let $[Y_1, Y_2] = Y_1Y_2 - Y_2Y_1$ be the *Lie bracket* of Y_1 and Y_2. If we extend Y_1 and Y_2 to right-invariant vector fields on G, this coincides with the Lie bracket from differential geometry.

Exercise 2.8. Fix a matrix group $G \subset GL(d,\mathbb{R})$ with abelian subgroup $A \subset G$ whose conjugation action is diagonalizable. Let E^α denotes the Lyapunov subspace for the conjugation action of A on T_eG with corresponding Lyapunov functional $\alpha : A \to \mathbb{R}$ (these are generalizations of those appearing in Definition 1.5). Show that if $Y_1 \in E^\alpha$ and $Y_2 \in E^\beta$, then $[Y_1, Y_2] \in E^{\alpha+\beta}$ (this is often shortened to $[E^\alpha, E^\beta] \subset E^{\alpha+\beta}$).

We use the following characterization of subgroups of G, which you may know from a Lie theory course

Lemma 2.7. *Let $G \subset GL(d,\mathbb{R})$ be a matrix Lie group, and $\mathfrak{u} \subset T_eG$ be a vector subspace. Then $\mathfrak{u}$ is tangent to an immersed subgroup if and only if for all $Y_1, Y_2 \in \mathfrak{u}$, $[Y_1, Y_2] \in \mathfrak{u}$.*

Combining this lemma with the preceding exercise immediately shows that for homogeneous actionsthe coarse Lyapunov distributions $E^{[\alpha]}$ are tangent to subgroups U^α, and the corresponding coarse Lyapunov foliation of G/Γ is the orbit foliation of the left action of U^α. If we assume the action is totally Cartan, then $\dim(U^\alpha) = 1$, so it must be isomorphic to $\mathbb{R}$. We can therefore write $U^\alpha = \{u_\alpha(s) : s \in \mathbb{R}\}$, with $u_\alpha(s_1)u_\alpha(s_2) = u_\alpha(s_1 + s_2)$. In this case, the generating vector field of the action of U^α belongs to $E^{[\alpha]} = E^\alpha$, and we get that for every $a \in A$:

$$au_\alpha(s) = u_\alpha(e^{\alpha(a)}s)a \tag{2.1}$$

> For a homogeneous Anosov action, the coarse Lyapunov foliations are orbit foliations of subgroups of G, called coarse Lyapunov subgroups. These actions are normalized by the A-action by expanding automorphisms for Anosov elements. When the action is Cartan, the coarse Lyapunov subgroups are isomorphic to $\mathbb{R}$, and the automorphisms given by (2.1).

3. Rank one factors & Rigidity

We saw in the previous lecture that hyperplanes are important for higher-rank, totally Anosov abelian group actions since they correspond to coarse Lyapunov distributions. In the homogeneous examples, they are exactly kernels of Lyapunov functionals. Here is another condition in which functionals on $\mathbb{R}^k$ naturally appear.

Definition 3.1. If $\mathbb{R}^k \curvearrowright X$ is a locally free action, a *(C^∞-)rank one factor* of the action is a locally free flow $\mathbb{R} \curvearrowright Y$, a C^∞ submersion $\pi : X \to Y$ and a homomorphism $\sigma : \mathbb{R}^k \to \mathbb{R}$ such that

$$\pi(a \cdot x) = \sigma(a) \cdot \pi(x).$$

Notice that by definition $\ker \sigma$ factors through the trivial action on Y. It therefore can't be transitive on X, assuming Y is not the 1-point space. Rank one factors are known to be a major obstruction to rigidity in the higher rank setting, since they have many ways to perturb out of a C^∞ conjugacy class.

Exercise 3.1. Show that if $\mathbb{R}^k \curvearrowright X$ is a totally Cartan action, then every 3-dimensional rank one factor is an Anosov flow on a 3-manifold.

Exercise 3.2. If $\mathbb{R}^k \curvearrowright (X, \mu)$ is an ergodic, measure-preserving action, a rank-one factor is any measurable factor (Y, ν) of X for which $\mathbb{R}^k$ factors through a ν-preserving flow via a homomorphism σ. Show that $\mathbb{R}^k \curvearrowright (X, \mu)$ has a nontrivial measurable rank one factor if and only if there exists a hyperplane $H \subset \mathbb{R}^k$ such that the restricted action $H \curvearrowright (X, \mu)$ is *not* ergodic. [*Hint*: Consider the ergodic decomposition of μ with respect to H]

3.1. **Rank one factors and transitivity of restrictions.** A recent breakthrough establishes the following smooth analogue of Exercise 3.2:

Theorem 3.2 (Spatzier-Vinhage)**.** *If $\mathbb{R}^k \curvearrowright X$ is a totally Cartan action with a transitive one-parameter subgroup $\{ta\} \subset \mathbb{R}^k$ and no circle factors, there exists a rank one factor if and only if there exists a hyperplane $H \subset \mathbb{R}^k$ without a dense orbit.*

Remark 3.3. There is a more precise version of this theorem which deals with the presence of circle factors are generalizes the transitivity condition.

We will not comment on the proof of this theorem here but describe the general strategy. One direction is clear, so the difficult part is showing that if you have a hyperplane without a dense orbit, you can build a nontrivial rank one factor. One may show through shadowing arguments that if H is not a coarse Lyapunov hyperplane, then it has a dense orbit. So we may assume that H is a coarse Lyapunov hyperplane. The idea is to show that the H-orbit closures all intersect a (k+2)-manifold on which H acts by $\mathbb{T}^{k-1}$ in finitely many H-orbits, and that the factor of this manifold by H is a 3-manifold equipped with an Anosov flow. The difficulty is showing this fact, as well as the H-orbit closures (at reasonably good points) build a C^∞ foliation of X.

3.2. **Building exact metrics and parameterizing flows.** So, assuming no rank one factors, we know that *every* hyperplane acts with a dense orbit. Here is another very useful lemma

Lemma 3.4 (Kalinin-Spatzier)**.** *If $\mathbb{R}^k \curvearrowright X$ is a totally Cartan action, E is a coarse Lyapunov distribution, $a \in \mathbb{R}^k$ belongs to the correspond neutral hyperplane, and $x \in X$, then*

$$|\log ||da|_E||| < d(a \cdot x, x)^\beta$$

for some $\beta > 0$ and whenever $d(a \cdot x, x)$ is sufficiently small and a is sufficiently large and far away from all other Lyapunov hyperplanes.

Remark 3.5. Constants have been dropped here in the interest of brevity and clarity, but an essential part of the application of this Lemma is knowing the rates. The lemma is proved by establishing a version of the Anosov closing lemma in higher-rank with very precise estaimates for elements near the hyperplane action. The proof heavily relies on the Cartan assumption.

The key use in the Lemma is combining it with the transitivity of the hyperplane action. These two features combined allow us to produce a Hölder metric which is invariant under H by extending a metric defined to be isometric along a dense H-orbit to its closure. And since H has a dense orbit, such a metric will be unique. Hence, each element of $\mathbb{R}^k$ must expan or contract it by a constant amount. In particular, we get the following

Proposition 3.6. *If $\mathbb{R}^k \curvearrowright X$ is a totally Cartan action with a transitive element and no rank one factor, then for each coarse Lyapunov distribution E, there exists a functional $\alpha : \mathbb{R}^k \to \mathbb{R}$ and a Hölder metric $||\cdot||_E$ defined on the bundle E such that*

$$||da(v)||_E = e^{\alpha(a)} ||v||_E \tag{3.1}$$

for all $a \in \mathbb{R}^k$.

Exercise 3.3. Prove Proposition 3.6 using the strategy outlined in the paragraph preceding it.

4. Building a homogeneous structure through path groups

From the previous section, if $\mathbb{R}^k \curvearrowright X$ is a totally Cartan action with a transitive element and no rank one factors, then we have the following assumptions, which we will call the fundamental assumptions:

> I. The restriction of the action to any hyperplane has a dense orbit.
> II. There exist Hölder metrics on each coarse Lyapunov distribution satisfying (3.1).

Let's assume that each coarse Lyapunov foliation is orientable, which can be achieved by passing to a double cover of X (we may need to do this several, but finitely many times). If W^α is a coarse Lyapunov foliation with functional α, define the flow $\eta^\alpha_s : X \to X$ so that $\eta^\alpha_s(x)$ is the point at distance s from x along the foliation W^α using the metric from II., positively oriented from x when $s > 0$ (similarly for $s < 0$). Since the metric used to define this action satisfies (3.1), the actions satsify the following condition which is identical to (2.1) in this setting:

$$a \circ \eta^\alpha_s = \eta^\alpha_{e^{\alpha(a)}s} \circ a. \tag{4.1}$$

> *The Big Idea*: In the homogeneous setting, the groups A and U^α generate G. If we can build relations among the flows η^α_s and the $\mathbb{R}^k$-action which resemble those of the examples, we can rebuild a G-action which is transitive (in the group-theoretic sense) on X, so X will be a homogeneous space!

4.1. The path group.

Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ denote the set of coarse Lyapunov functionals/distributions, and consider the group

$$\mathcal{P} = \mathbb{R}_{\alpha_1} * \cdots * \mathbb{R}_{\alpha_n} * \mathbb{R}^k,$$

the free product of n copies of $\mathbb{R}$, and one copy of $\mathbb{R}^k$. The elements of $\mathcal{P}$ are words in these groups. We denote an element of $\mathbb{R}_{\alpha_i}$ by $s^{(\alpha_i)}$, where $s \in \mathbb{R}$. A sample element of $\mathcal{P}$ might look like

$$\rho = s_1^{(\alpha_2)} * s_2^{(\alpha_7)} * s_3^{(\alpha_3)} * a * s_4^{(\alpha_2)}$$

The only relations among the group is the ability to combine adjacent letters coming from the same group, and that the identity for all groups making up the free product is the common identity element for $\mathcal{P}$. For instance, the word above can't be reduced, but we get the following relation

$$s_1^{(\alpha_2)} * s_2^{(\alpha_3)} * (-s_2)^{(\alpha_3)} * s_3^{(\alpha_2)} = (s_1 + s_3)^{(\alpha_2)}$$

Here is the trick: $\mathcal{P}$ acts on X! Given a word in the groups, we apply its letters from right to left, where $s^{(\alpha_i)}$ acts via the flow $\eta^{\alpha_i}_s$ and a acts through the $\mathbb{R}^k$-action. We call this the path group, since we can trace out the action of an element $\mathcal{P}$ by drawing paths along the corresponding foliations.

Exercise 4.1. Show that if $\mathcal{C}_x = \mathrm{Stab}_{\mathcal{P}}(x)$ is the stabilizer of x under the action of $\mathcal{P}$, then $\mathcal{C}_x$ consists of cycles based at x. Furthermore, show that in the case of a homogeneous action, there is a homomorphism $\pi : \mathcal{P} \to G$ uniquely defined by sending $s^{(\alpha_i)}$ to $u_{\alpha_i}(s)$ and a to a, and that for every $x \in X$, $\mathcal{C}_x = \ker \pi$.

Here is the magic lemma which makes this method work. We use H° to denote the identity component of a group H which may not be connected.

Lemma 4.1. *Let $\mathcal{P} \curvearrowright X$ be the continuous action of a free product of Lie groups on a topological manifold. If $\mathcal{C}_x^c = \mathrm{Stab}_{\mathcal{P}}(x)^\circ$ does not depend on $x \in X$, then $G := \mathcal{P}/\mathcal{C}_x^c$ is Lie group acting on X, and if the $\mathcal{P}$-action is transitive (in the group-theoretic sense), X is a G-homogeneous spaces.*

Sketch of Proof. We first show that $\mathcal{C}^c$, the common stabilizer subgroup, is normal in $\mathcal{P}$. Observe that if $\rho \in \mathcal{P}$ and $\sigma \in \mathcal{C}^c$, then $\rho^{-1}\sigma\rho \cdot x = \rho^{-1}\sigma \cdot (\rho \cdot x) = \rho^{-1}\rho \cdot x = x$ (here we use that σ stabilizes $\rho \cdot x$). We come to the question of topology on $\mathcal{P}$. A group topology exists for which the action is continuous, but it is terrible (not metrizable, for instance). The good news is that $\mathcal{C}^c$ is a closed subgroup, since it is a stabilizer, so $G = \mathcal{P}/\mathcal{C}^c$ is a group which acts on X.

One now must show that G is Lie. This is a careful game in studying classical theorems in Lie criteria. The main issue is the non-local compactness of $\mathcal{P}$. It is tempting to say that the evaluation map is a homeomorphism between G and the universal cover of X, but this is not guaranteed. We skip this discussion here for brevity. □

If one can show that stabilizer (cycle) subgroups for a free product are independent of the basepoint, then the free product action factors through a Lie group action.

4.2. **Symplectic Cycles.** In this section, we aim to show that the free product $\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}$ of any pair of negatively proportional Lyapunov exponents factors through a Lie group action. The main tool is the following observation.

Lemma 4.2. *Suppose that $\sigma \in \mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}$ fixes x, and $a \in \ker \alpha$. Then σ fixes $a \cdot x$.*

Proof. By (4.1), if $\sigma = t_1^{(\alpha)} * s_1^{(-c\alpha)} * \cdots * t_\ell^{(\alpha)} * s_\ell^{(-c\alpha)}$, then $a\sigma = \sigma a$. In particular, $\sigma \cdot (a \cdot x) = a \cdot (\sigma \cdot x) = a \cdot x$. One can represent this geometrically by imagining σ as a cycle, and "sliding" the cycle along a $\ker \alpha$-orbit. □

The proof of the following requires some advanced tools from Lie criteria, so we omit it, but follows the scheme described in the previous section: once a stabilizer is independent of x, the action of a free product factors through a Lie group action.

Corollary 4.3. *If $x_0 \in X$ has a dense $\ker \alpha$-orbit, $\mathcal{C}^{\alpha,-c\alpha} = \mathrm{Stab}_{\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}}(x_0)$ is a closed normal subgroup of $\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}$, and $G_\alpha := (\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha})/\mathcal{C}^{\alpha,-c\alpha}$ is a Lie group acting on X.*

One can actually do better for symplectic cycles, we can describe exactly what groups can appear as G_α. We know, since it is a factor of $\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}$, that it is generated by two one-parameter subgroups. Let $Y_\alpha, Y_{-c\alpha} \in \mathrm{Lie}(G_\alpha) = T_eG_\alpha$ be generators of these subgroups.

Since the $\mathbb{R}^k$-action preserves the set of cycles, the automorphism of $\mathbb{R}_\alpha * \mathbb{R}_{-c\alpha}$ descends to an automorphism of G_α with Y_α and $Y_{-c\alpha}$ as eigenspaces at T_eG_α. As in the homogeneous case, we can show that if $Z = [Y_\alpha, Y_{-c\alpha}]$, then Z is an eigenvector of eigenvalue $e^{\alpha(a)-c\alpha(a)} = e^{(1-c)\alpha(a)}$. It therefore cannot equal Y_α or $Y_{-c\alpha}$, and has behavior determined by the functional $(1-c)\alpha(a)$. This contradicts Cartan, unless $c = 1$, in which case it must be in the orbit direction. Continuing this analysis yields the following classification:

Theorem 4.4. *The group G_α is locally isomorphic to $SL(2,\mathbb{R})$,* Heis *(the 3-dimensional Heisenberg group), or $\mathbb{R}^2$. In the first two cases, $c = 1$.*

The main goal of this work is to conclude the following ability to commute elements:

Corollary 4.5. *For sufficiently small $t, s \in \mathbb{R}$, there exists $\bar{t}, \bar{s} \in \mathbb{R}$ and $a \in \mathbb{R}^k$ such that*

$$t^{(\alpha)} * s^{(-c\alpha)} = \bar{s}^{(-c\alpha)} * \bar{t}^{(\alpha)} * a \pmod{\mathcal{C}_{\text{const}}}$$

where $\mathcal{C}_{\text{const}}$ is a closed, normal group of cycles which does not depend on the basepoint.

4.3. **Commutator Cycles.** We wish to establish a result similar to Corollary 4.5, but for linearly independent weights α and β. When α and β are linearly independent, this can actually be done by analyzing group commutators.

Exercise 4.2. If $\rho = [s^{(\beta)}, t^{(\alpha)}] := (-s)^{(\beta)} * (-t)^{(\alpha)} * s^{(\beta)} * t^{(\alpha)}$ and a is an Anosov element such that $\alpha(a), \beta(a) < 0$, show that $\rho \cdot x \in W_a^s(x)$. Conclude that $\rho \cdot x$ is in a common stable manifold whose tangent distribution is

$$E^{(\alpha,\beta)} := \bigoplus_{\substack{\gamma = p\alpha + q\beta \\ p,q>0}} E^\gamma$$

Hint: This is a linear algebra exercise in disguise. Show that weights not of the form in the equation above can be expanded while keeping α and β contracting. Then find another argument to rule out α and β from appearing.

For these notes, let us deal onlywith the case when there is only one exponent γ of the form $\gamma = p\alpha + q\beta$, $p, q > 0$. With careful bookkeeping and more nuanced arguments, other cases can be addressed. In this simplified setting, we can define a unique function $\rho^{\alpha,\beta}(s,t,x) = u$ to be the unique number such that $[s^{(\beta)}, t^{(\alpha)}] \cdot x = u^{(\gamma)} \cdot x$. The following equations can be deduced from drawing pictures, or through algebraic manipulations (hopefully to be added to these notes later...):

$$e^{\gamma(a)} \rho^{\alpha,\beta}(s,t,x) = \rho^{\alpha,\beta}(e^{\alpha(a)}s, e^{\beta(a)}t, a \cdot x) \tag{4.2}$$

$$\rho^{\alpha,\beta}(s_1 + s_2, t, x) = \rho^{\alpha,\beta}(s_1, t, x) + \rho^{\alpha,\beta}(s_2, t, \eta^\alpha_{s_1}(x)) \tag{4.3}$$

While (4.2) basically follows from (4.1), (4.3) requires the following observation: since γ is the unique exponent appearing in $E^{(\alpha,\beta)}$, $E^{(\alpha,\gamma)}$ and $E^{(\beta,\gamma)}$ are both trivial (another Exercise!), so η^γ commutes with both η^α and η^β.

An important aspect of our analysis will rely on the fact that $p \geq 1$ (recall $\gamma = p\alpha + q\beta$).

Lemma 4.6. *if $\rho^{\alpha,\beta}(s,t,x) \not\equiv 0$, then either $p \geq 1$ or $q \geq 1$.*

The proof of this lemma requires some faimilarity with regularity of "fast foliations" in unstable foliations (ie, foliations to bundles inside the stable bundle which are the faster of two in a dominated splitting).

Sketch of Proof. Assume that $q < 1$ and $q < 1$. We will show that $\rho^{\alpha,\beta}(s,t,x) \equiv 0$. Indeed, let $x' = [t^{(\alpha)}, s^{(\beta)}] \cdot x = \rho^{\alpha,\beta}(s,t,x)^{(\gamma)} \cdot x$. Choose a such that $\alpha(a) = -1$, and $\beta(a) < 0$ is very small in absolute value so that$\alpha(a) < \gamma(a) < \beta(a) < 0$. Then W^α is C^∞ in $W^{[\alpha,\beta]}$, since it is the fastest contracting foliation. By reversersing the roles of α and β, we conclude that $W^{\beta,\gamma}$, the foliation tangent to $E^\beta \oplus E^\gamma$ is C^∞ as well.

Notice that the points $t^{(\alpha)} \cdot x$ and $s^{(\beta)} * t^{(\alpha)} \cdot x$ are the images of x and $s^{(\beta)} * \rho^{\alpha,\beta}(s,t,x)^{(\gamma)} \cdot x$ under the α-holonomy, which we showed above is C^∞. Hence we get a Lipschitz estimeate:

$$1/L \leq \frac{d(t^{(\alpha)} \cdot x, s^{(\beta)} * t^{(\alpha)} \cdot x)}{d(x, s^{(\beta)} * \rho^{\alpha,\beta}(s,t,x)^{(\gamma)} \cdot x)} \leq L \tag{4.4}$$

for some $L > 1$. Now, choose an element b such that $\beta(b) < 0$, but $\alpha(b)$ is negative but very close to 0, in such a way that $\beta(b) < \gamma(b) < \alpha(b) < 0$. Applying the element b to (4.4) gets us a contradiction, since the numerator will go to 0 at rate $e^{n\beta(b)}$, but the denominator will go to 0 at a slower rate, $e^{n\gamma(b)}$ *unless* $\rho^{\alpha,\beta}(s,t,x) = 0$. This is what we aimed to prove. □

Let us without loss of generality assume that $p \geq 1$, since by Lemma 4.6, either $p \geq 1$, $q \geq 1$ or $\rho^{\alpha,\beta} \equiv 0$ (in the last case, we are done!).

Lemma 4.7. *If $\rho^{\alpha,\beta} \not\equiv 0$, $p = 1$. Furthermore, there exists some $m \in \mathbb{R}$ such that $\rho^{\alpha,\beta}(s,t,x) = ms$ or every $s,t \in \mathbb{R}$ and $x \in X$.*

Sketch of Proof. Let's introduce a new function as shorthand for $\rho^{\alpha,\beta}$, suppressing some notation and arguments:

$$\varphi(s,x) = \rho^{\alpha,\beta}(s,1,x)$$

Our aim is to show that $\varphi(s,x)$ is linear, and has slope independent of x. A remark before beginning: it is tempting to apply (4.2) to φ, and we can as long as we apply only elements of $\ker\beta$ (since we must preserve that the second component is 1).

First, observe that the cocycle property (4.3) implies that φ is locally Lipschitz in s. Indeed, when $|s-t| \leq 1$.

$$|\varphi(s,x) - \varphi(t,x)| = |\varphi(s-t, \eta_t^\alpha(x))| \leq |s-t|^u \, |\varphi(\pm 1, a \cdot \eta_t^\alpha(x))| \leq m_0 \, |s-t|$$

where $m_0 = \sup_{x \in X} \varphi(\pm 1, x)$ and a is chosen so that $\beta(a) = 0$ and $\alpha(a) = \log|s-t|$. Here we have also used (4.2) and the fact that the $u \geq 1$. Since φ is locally Lipschitz in s, it has a derivative in s almost everywhere for a fixed x. Define

$$f(x) = \left.\frac{d}{ds}\right|_{s=0} \varphi(s,x) = \lim_{s \to 0} \frac{1}{s}\varphi(s,x)$$

wherever it exists. Since it exists for Lebesgue almost-every point of *every* leaf $W^\alpha(x)$, f exists on a dense subset. Observe that if $f(x)$ exists, then so does $f(b \cdot x)$ for $b \in \ker\beta$. Indeed,

$$f(b \cdot x) = \left.\frac{d}{ds}\right|_{s=0} \varphi(s, b \cdot x) = \left.\frac{d}{ds}\right|_{s=0} e^{u\alpha(b)}\varphi(e^{-\alpha(b)}s, x) = e^{(u-1)\alpha(b)} f(x) \tag{4.5}$$

by the chain rule. Now, we know that $f \leq m_0$ since m_0 is a Lipschitz constant for φ near $s = 0$. If $u > 1$, then by choosing some $b \in \ker\beta$ with $\alpha(b)$ large, we get that $f(b \cdot x)$ can become arbitrarily large unless $f \equiv 0$. Hence either $f \equiv 0$ wherever it exists, or $u = 1$.

We now claim that f is constant on W^α-leaves whenever defined. If $f \equiv 0$ when it exists, we are done, otherwise, we have $u = 1$. In particular, it follows from (4.5) that f is $\ker\beta$-invariant.

Now, consider a point x at which f exists, choose some $a \in \mathbb{R}^k$ such that $\beta(a) = 0$ and $\alpha(a) > 0$. Then let n_k be a subsequence for which $a^{n_k} \cdot x$ converges to some point z. Since W^α is contracted by a, it follows that $a^{n_k} \cdot y$ also converges to z. Hence

$$f(x)=\lim_{s\to 0}\frac{1}{s}\varphi(s,x)=\lim_{k\to\infty}e^{n_k\alpha(a)}\varphi(e^{-n_k\alpha(a)},x)=\lim_{k\to\infty}\varphi(1,a^{n_k}\cdot x)=\varphi(1,z)$$

Hence $f(x)=\varphi(1,z)$ at any point z in the ω-limit set of x under any a which expands α. Since $\ker\beta$ has a dense orbit (and there is a lot of recurrence due to the prevalence of $\mathbb{R}^k$-periodic orbits), there exists a point x for which the set of such limits is everything. Hence, since the ω-limit set of any point is always nonempty, f is constant wherever it exists.

Now, since it exists for Lebesgue-almost every point in each W^α, leaf, it follows that $\partial_s\varphi(s_0,x)=f(\eta^\alpha_{s_0}(x))$ is constant and equal to some m (prove this using (4.3)). Therefore, $\varphi(s,x)=ms$ is linear. □

Now, to finish the argument, note that if $\psi(t,x)=\rho^{\alpha,\beta}(1,t,x)$, then by (4.2) and using Lemma 4.7, if $a\in\mathbb{R}^k$,

$$\begin{aligned}\psi(t,a\cdot x)=\rho^{\alpha,\beta}(s,t,a\cdot x)&=e^{p\alpha(a)+q\beta(a)}\rho^{\alpha,\beta}(e^{-\alpha(a)}s,e^{-\beta(a)}t,x)\\&=e^{q\beta(a)}\rho^{\alpha,\beta}(s,e^{-\beta(a)}t,x)=e^{q\beta(a)}\psi(e^{\beta(a)}t,x)\end{aligned}$$

So in this case, we get an intertwining property for all $a\in\mathbb{R}^k$, not just $a\in\ker\alpha$. So we may apply elements of $\ker\beta$, which has a dense orbit, and conclude that $\psi(t,\cdot)$ is constant everywhere on X.

Exercise 4.3. Use the fact that φ and ψ are independent of x, together with (4.3) (and an analogous property for the second component, whose deduction we leave to you), to show that there exists a constant m, independent of x, such that

$$\rho^{\alpha,\beta}(s,t,x)=mst$$

Remark 4.8. The reason for the particularly simple form of the commutator comes from the fact that there is only one γ between α and β. For the general case, one sets up a complicated induction, and the property (4.3) has a nontrivial polynomial term coming from the fact that while η^γ may not commute with η^α and η^β, the relations are well-understood polynomials from a previous induction step.

The main conclusion of this section is actually simply the following (cf Corollary 4.5):

Corollary 4.9. *For any $s,t\in\mathbb{R}$ and linearly independent Lyapunov exponents α,β,*

$$t^{(\beta)}*s^{(\alpha)}=\bar{s}^{(\alpha)}*\bar{t}^{(\beta)}*\bar{\rho}=\hat{\rho}*\hat{s}^{(\alpha)}*\hat{t}^{(\beta)}\pmod{\mathcal{C}_{\mathrm{const}}}$$

for some $\hat{t},\hat{s},\bar{t},\bar{s}\in\mathbb{R}$ and $\hat{\rho},\bar{\rho}\in\mathcal{P}$, words whose letters come only from weights of the form $p\alpha+q\beta$, $p,q\in\mathbb{Z}_+$. Here again, $\mathcal{C}_{\mathrm{const}}$ is a closed, normal group of cycles which does not depend on the basepoint.

4.4. Reduced Forms and the Quadrant Argument.

Fix an Anosov a, and let $\{\alpha_1,\dots,\alpha_m\}$ denote the set of Lyapunov exponents which have positive evaluation on a, and $\{\beta_1,\dots,\beta_n\}$ denote the set of Lyapunov exponents with negative evaluation. We list them in a ciruclar ordering, which we will define later this section, meaning that products of elements in this circular ordering provide natural coordinates on the stable and unstable manifold, respectively. The main goal of this section is to prove:

Theorem 4.10. *If $\rho \in \mathcal{P}$ is any word in the path group, then modulo $\mathcal{C}_{\text{const}}$, a closed, normal subgroup stabilizing every point of X, we may express ρ uniquely as*

$$\rho = t_1^{(\alpha_1)} * \cdots * t_m^{(\alpha_m)} * s_1^{(\beta_1)} * \cdots * s_n^{(\beta_n)} * a$$

Corollary 4.11. *$G := \mathcal{P}/\mathcal{C}_{\text{const}}$ is Lie, X is a G-homogeneous space, and the $\mathbb{R}^k$-action is a translation action in the coordinates provided by G.*

5. Other examples and exercsies

5.1. **Suspensions of lattice actions.** The suspension of a diffeomorphism to a flow has a natural generalization to group actions. Let $\Gamma \subset G$ be a lattice in G, and $\Gamma \curvearrowright X_0$ be an action of Γ on a manifold X_0. Then let $\tilde{X}$ be the product space $G \times X_0$, and let Γ act on right on X_0 via

$$(g, x) \cdot \gamma = (g\gamma, \gamma^{-1} \cdot x).$$

Since Γ is discrete in G, this action is C^∞ and properly discontinuous and the quotient space $X = \tilde{X}/\Gamma$ is a C^∞ manifold. The *suspension action* of G on X is the left translation action

$$g \cdot (g_0, x)\Gamma = (gg_0, x).$$

Exercise 5.1. Show that if $G = \mathbb{R}$ and $\Gamma = c\mathbb{Z}$ for some $c > 0$, this is smoothly conjugate to the usual suspension construction with constant roof function c.

5.2. **Abelian Examples.** We review examples of abelian actions of the type we wish to consider. We put the basic definitions here for reference, and establish properties later.

Example 5.1. Let $G = SL(d, \mathbb{C})$ and $\Gamma \subset H$ be a cocompact lattice. If

$$A = \left\{ \operatorname{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_d}) : \sum t_i = 0 \right\} \cong \mathbb{R}^{d-1}$$

is as above (this is the split Cartan subgroup) and $K = \left\{ \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots e^{i\theta_d}) : \sum \theta_i = 0 \right\}$, then the translation action of A on $K\backslash G/\Gamma$ is Anosov.

Example 5.2. Let $A_1, A_2 \in SL(d, \mathbb{Z})$ be commuting, hyperbolic matrices, and assume that they are images under the exponential map of some $Y_i \in \mathfrak{sl}(d, \mathbb{R})$ (this happens whenever the eigenvalues are all positive). Wite $A_i = e^{Y_i}$, and ${A_i}^t = e^{tY_i}$, allowing us to take non-integer powers of A_i. Finally define the group $H = \mathbb{R}^2 \ltimes \mathbb{R}^d$ as a semidirect product of $\mathbb{R}^2$ with $\mathbb{R}^d$, where if $a, b \in \mathbb{R}^2$ and $v, w \in \mathbb{R}^d$, the semidirect product structure is given by

$$(a, v) \cdot (b, w) = (a + b, {A_1}^{b_1} {A_2}^{b_2} v + w)$$

Notice that since each A_i is a matrix with integer entries, the set of integer points in H is a cocompact lattice $\Gamma = \mathbb{Z}^2 \ltimes \mathbb{Z}^d$.

Exercise 5.2. Show that the action in Example 5.2 is C^∞ conjugate to the suspension of the $\mathbb{Z}^2$-action on $\mathbb{T}^d$ generated by A_1, A_2. [*Hint*: Show that the set $\left\{(0, v) : v \in \mathbb{R}^d\right\}\Gamma$ is a torus transverse to $\mathbb{R}^2$, then find its stabilizer and the induced maps by left translations. Use this to builid a conjugacy directly.]