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An isometric embedding problem arising from general relativity

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Abstract. We consider an isometric embedding problem that arises from general relativity. Physicists have sought to find embedding diagrams, isometric embeddings into Euclidean three space, of slices of initial data for Einstein's equations. One example is Misner's wormhole universe. It is a complete and negatively curved but doesn't satisfy either Efimov's nonexistence nor Hong's existence conditions for isometric immersibility. We describe some results for nonpositively curved surfaces under additional extrinsic hypotheses. In joint work with H.G. Chan we show that a complete, one ended, nonflat, nonpositively curved surface embedded in Euclidean three space with square integrable second funamental form must lie a fixed distance from a plane and have continua of parabolic points heading to infinity. This implies Chan's result that the Misner surface doesn't admit such an embedding diagram.

We describe some results about complete nonpositively curved surfaces embedded in Euclidean three space. The motivation comes from an isometric embedding quesion posed by general relativists, who are interested in finding embedding diagrams of interesting initial manifolds for the evolution problem for Einstein's equations. We describe embedding diagrams and illustrate in the case of Schwarzschild space. We describe Misner's wormhole manifold which is compatible with the Einstein equations. Existing isometric embedding theorems don't apply to it. Under additional extrinsic hypotheses we are able show geometric properties of complete nonpositively curved surfaces. We prove that a complete, one ended, nonflat, nonpositively curved surface embedded in $\mathbf{R}^3$ with square integrable second fundamental form lies a finite distance from a plane. From this we recover Chan's theorem that no embedding diagram exists for the Misner surface under reasonable extrinsic hypotheses. A detailed account of these results will appear elsewhere[CTr].

The initial value problem for general relativity is to specify a complete Riemannian three manifold $x \in (M^3, g)$

$$g = \sum_{i,j=1}^{3} g_{ij}(x) dx^i \, dx^j$$

and its second fundamental form h_{ij} and to evolve them into a 4-dimensional space-time $(t,x) \in (N^4, g^4)$ using Einstein's source free equations (no matter). The initial manifolds are assumed to be symmetric about $t = 0$,

$$g^4 = -dt^2 + g \qquad \text{at } t = 0,$$

and *momentarily stationary*

$$\left.\frac{dg}{dt}\right|_{t=0} = 0.$$

The initial surface has to satisfy Einstein's equations. The equations of Einstein's system not involving second time derivatives of the metric say that in this case (M, g) has to be *scalar flat.* [Mi] The rest of the equations specify second time derivatives of the metric,and thus the evolution.

Misner's surface. Of particular interest is the *collision of black holes* which has been extensively studied numerically[AP]. Misner's data and Brill & Lindquist[BL] data are examples of three manifolds with wormholes. We describe Misner's construction[Mi]. To start, assume topologically $M = \Sigma \times \mathbf{S}^2 \ni (\mu, \theta, \phi)$ and look for metrics conformal to the canonical one

$$g = \psi^4(d\mu^2 + d\theta^2 + \sin^2\theta \, d\phi^2).$$

Scalar curvature is given by

$$L\psi = \Delta_\Sigma \psi - \tfrac{1}{4} R_\Sigma \psi = c R_g \psi^5$$

with $R_\Sigma = 1$. One solution with $R_g = 0$ is (gotten by pulling back the flat plane metric to $\mathbf{R}^2$ via elliptical coordinates)

$$\psi = \frac{1}{\sqrt{\cosh\mu - \cos\theta}},$$

so $L\psi = 0$. The conformal metric blows up at $\mu = 0, \theta = 2\pi k$. Also near (0,0),

$$\psi^4 \sim \frac{1}{(\mu^2 + \theta^2)^2}$$

so the metric is asymptotic to a flat end. It is made periodic in μ using superposition

$$\psi = \sum_{-\infty}^{\infty} \frac{1}{\sqrt{\cosh(\mu + 2na) - \cos\theta}}.$$

Take quotient by translation $(\mu, \theta) \mapsto (\mu + 2ka, \theta))$. The result is a scalar flat metric on $\mathbf{S}^1 \times \mathbf{S}^2 - k$ pts.. The *Misner surface* is the slice

$$(\mathrm{MS}_k) \qquad \Big(\mathbf{S}^1 \times \mathbf{S}^1 - \{(4ak, 0)\}, \psi^4(d\mu^2 + d\theta^2)\Big).$$

The usual picture of a wormhole slice (MS_1) is the surface given by a handle connected to an asymptotically flat plane.

Embedding Diagrams. Consider the surface corresponding to the ϕ = const. slice. An *embedding diagram* is an isometric embedding of this surface into $\mathbf{R}^3$. An embedding diagram for the Schwarzschild spacetime (one stationary black hole) is easily found. It is what's usually pictured as two asymptotically flat planes connected by a tube. The Schwarzschild metric is independent of t. In the usual coordinates it is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \dfrac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

$r = 2m$ is a coordinate singularity. Take the slice corresponding to t = const. and ϕ = const. To find an isometric embedding, make the Ansatz that the position vector takes the form

$$X(r,\theta) = (a(r), b(r)\cos\theta, b(r)\sin\theta).$$

Equating the pullback of the Euclidean metric by X to the Schwarzschild metric gives ODE's for the isometric embedding. Integrating yields

$$X(r,\theta) = \left(\sqrt{8m}\sqrt{r-2m}, r\cos\theta, r\sin\theta\right).$$

This is the usual parabola of revolution picture of the Schwarzschild universe. One checks that the embedding continues beyond the coordinate singularity at $r = 2m$. The second fundamental form $h_{ij} \in L^2$ and the curvature satisfies

$$K = \frac{-m}{r^2(2-2m)}, \qquad h_{ij} = \begin{pmatrix} -\frac{\sqrt{2m}}{2r^{3/2}} & 0 \\ 0 & \frac{\sqrt{2m}}{\sqrt{r(r-2m)}} \end{pmatrix}$$

The question that motivates our study is: can one find an embedding diagram for the Misner surface (MS_1)? This surface has[RP]

$$K \sim \frac{1}{\mathrm{dist}(x, x_0)^3}, \qquad K < 0.$$

An interesting numerical effort was made to consturct an embedding diagram for (MS_1). Physicists assumed that if an embedding existed, it would concievably behave at infinity like the embedding of the Schwarzschild space. By choosing constants to make the masses of (MS_1) and Schwarzschild equal so that they appear similar from a distance, Romano and Price[RP] attempted to propogate from the flat end into the torus starting from Schwarzshild data. They numerically solve the *hyperbolic* Darboux equation for the second fundamental form and integrate the Gauss and Codazzi equations to get the surface. It turns out that the numerical procedure develops a shock (the asymptotic directions coincide) before too much of the surface is reconstructed. This is inevitable and is reminiscent of Amsler's proof[Am] (that shocks must develop in any immersion) of Hilbert's theorem about the nonexistence of C^2 isometric immersions of the hyperbolic plane into $\mathbf{R}^3$. Romano and Price did observe an index argument which rules out the success of a nice solution.

Historical remarks. Poznyak-Shikin[PS] and Rozendorn[Rz] survey embedding problems for nonpositively curved sufaces. We mention some of the noexistence results for embedding surfaces with $K < 0$. The first is Hilbert's theorem[Hi] that there are no C^2 isometric immersions of the hyperbolic plane $\mathbf{H}^2 \subset \mathbf{R}^3$. Efimov[Ef1], [KM] showed that there are no C^2 isometric immersions of any complete $(\mathbf{R}^2, g) \subset \mathbf{R}^3$ s.t. $K \le -b^2 < 0$. Later he published the theorem[Ef2] that there are no C^2 isometric immersions of complete $(\mathbf{R}^2, g) \subset \mathbf{R}^3$ s.t. $K < 0$ and for all x, y,

$$\left| \frac{1}{\sqrt{-K(x)}} - \frac{1}{\sqrt{-K(y)}} \right| \le c_1 \mathrm{dist}(x, y)) + c_2.$$

The argument is extremely topological. Efimov partially succeeded to give a completely analytic argument[Ef3]: that there are no C^3 isometric immersions of complete $(\mathbf{R}^2, g) \subset \mathbf{R}^3$ s.t. $K < 0$ and for all x,

$$\left| \nabla \frac{1}{\sqrt{-K(x)}} \right| < \frac{\sqrt{2}}{3}.$$

Perelman has nonexistence results for general halfspaces[Pe].

If the curvature decays faster, then there are positive results. By integrating the hyperbolic Codazzi system, J.X. Hong[Ho] showed that $(\mathbf{R}^2, g)$ with g sufficiently smooth such that the curvature $K < 0$ decays like

$$\frac{\partial}{\partial \rho}(\log(|K|\rho^{2+\delta}) \le 0$$

as $\rho \gg 1$, where ρ is distance from a point, then M admits an isometric immersion into $\mathbf{R}^3$. Previously, the method was used to immerse pieces of negatively curved manifolds[Ka], [Po], [Rh], [Sh], [Tn].

Several results apply to surfaces satisfying also extrinsic conditions. We mention a result of Schoen & Simon[SS]: if $(\mathbf{R}^2, g)$ is complete with quadratic area growth, then $|\sum h_{ij}^2| \le -cK$ implies M is a plane. That the Misner's surface doesn't admit isometric embeddings can be seen by

Theorem. [Ch2] *Let M^2 be complete, oriented, one-ended $K \le 0$, $K \not\equiv 0$ such that parabolic points $\{x : K(x) = 0\}$ are isolated in M. Then M admits no C^2 isometric embedding with $h_{ij} \in L^2$. (Embeddedness near the end suffices.)*

The theorem is sharp in the sense there exist surfaces which son't satisfy one of the hypotheses. Examples are the flat surface (plane curve $\times \mathbf{R}$), Enneper's surface which is not embedded near the end and the catenoid which has two ends. Another example is thee solution of

$$(1 + z)x^2 - (1 - z)y^2 = 2z(1 - z^2)$$

in $\mathbf{R}^3$ which is topologically a handle attached to the plane. It satisfies all hypotheses except that the lines at $z = \pm 1$ have zero curvature so it has nonisolated parabolic points. For this surface

$$K \sim \sum h_{ij}^2 \sim \mathrm{dist}(p, p_0)^{-4} \qquad \text{as } p \to \infty.$$

Our main result shows that this situation is typical in this class of surfaces.

Slab theorem. The idea is that the two major hypotheses clash. The square integrability of the second fundamental form says that the ends of complete non-positively curved surface embedded in three space have to be very tame. However, the oscillation of the surface due to the presence of any negative curvature magnifies as it propogates to the end. An example of this is a theorem of Bernstein(1915) which was first completely proved by E. Hopf[Hp2].

Theorem. *Let $u \in C^2(\mathbf{R}^2)$ satisfy $u_{xx}u_{yy} - u_{xy}^2 \le 0$ and for some point "< 0". Then u has linear growth: there are constants $c, R_0 > 0$ so that for all $r > R_0$*

$$\sup_{|(x,y)|=r} u(x,y) - \inf_{|(x,y)|=r} u(x,y) \ge cr.$$

Idea of the proof. By the maximum principle, nodal domains $\{(x,y) : u(x,y) > 0\}$ are noncompact and if there is a nodal domain which lies in a sector of opening angle less than π, then u has linear growth in that sector. Then a topological argument is given to show that there must be such a nodal domain. Assume that u doesn't have linear growth. The local argument says that if at some point $K < 0$ then nearby is a point where $\nabla u \ne 0$ and $K < 0$. By rotation and translation arrange that the point is $(0,0)$ and $\nabla u(0,0) = (0,\xi)$ with $\xi > 0$. Let $\eta(x,y) = u(x,u) - \xi y$. η is also negatively curved and has four nodal domains touching zero. The global part says that by the growth hypothesis, η is negative not too far above and positive not too far below the x-axis. Viewing $\mathbf{R}^2$ as a disk and adding two points at infinity corresponding to $\pm\infty$ of the x-axis, then both $\pm\infty$ are accessible by Jordan curves in the positive and negative regions away from the axes. Finally, Hopf proves an accessibility statement[Hp1], that if the nodal domains touching zero aren't contained in a sector, then both $\pm\infty$ can be connected by Jordan arcs to any point in the nodal domain. This is a contradiction. $\square$

Away from a compact set our surfaces look like graphs. Our argument generalizes Hopf's to these surfaces.

Theorem. [CTr] *Let M be a smooth, complete, oriented, $K \le 0$, $K \not\equiv 0$, one ended surface which is C^2 immersed in $\mathbf{R}^3$ such that it is embedded near its end. Then M lies between two parallel planes. Moreover, choosing the planes as close together as possible, then there is a set of contact with both planes which is parabolic, and consists of continua heading to infinity. (e.g. lines, stars, strips, halfspaces.)*

Idea of the proof. (1.) By a theorem of B. White[Wh], complete, $K \le 0$ and $h_{ij} \in L^2$ implies that the Gauss map extends continuously across the end. Since we have assumed embeddedness near the end, the manifold must be a graph near its end. Near the end, for $|(x,y)| \ge R$ the surface can be parameterized $X = (x, y, u(x,y))$ so that $\nabla u \to 0$ as $(z,y) \to \infty$.
(2.) Generalize the local part of the argument of Hopf. Suppose that the narrowest nonvertical parallel planes which contain the $|(x,y)| \le R$ part of the surface don't already straddle the whole surface. Then after rotation, for some $c_1, c_2, \xi > 0$, there are at least two "nodal" domains $\{u(x,y) > c_1 + \xi y\}$ and at least two domains $\{(x,y) : u(x,y) < -c_2 + \xi y\}$ in the graph part of the surface near the end. The global part of the argument is the same as Hopf's. Hence u has linear growth. In particular there is a sequence $(x_i, y_i) \to \infty$ so that $\nabla u(x_i, y_i) \not\to 0$ which contradicts the first part.

(3.) By the second step, the narrowest parallel planes straddling the $|(x,y)| \le R$ part of the surface must touch it at interior points. By the maximum principle, touch points have $K = 0$ and the components of the complemenatry set in the planes are convex and non noncompact. The structure of the contact set follows.$\square$

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