

## The Strong Isoperimetric Inequality of Bonnesen

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Among all simple closed curves in the plane with a fixed length, the circle has the greatest area. The Isoperimetric Inequality, the mathematical statement of this fact, is done in two parts.

### Isoperimetric Inequality.

1. Let  $\gamma \subset \mathbf{E}^2$  be a simple closed curve in the plane whose length is  $L$  and that encloses an area  $A$ . Then the following inequality holds

$$(I) \quad 4\pi A \leq L^2.$$

2. If equality holds in (I), then the curve  $\gamma$  is a circle.

Simple means that the curve is assumed to have no self intersections. A circle of radius  $r$  has length  $L = 2\pi r$  and it encloses an area of  $A = \pi r^2 = \frac{L^2}{4\pi}$ . Thus if  $\gamma$  is a simple closed curve of length  $L$  and encloses an area  $A$ , the isoperimetric inequality says that  $\gamma$  encloses an area no bigger than the area of the circle with the same length, or (I).

The Isoperimetric inequality may be proved by any number of methods, including calculus of variations, integral geometry, complex analysis, geometric measure theory, symmetrization [Bl] and even by the heat flow PDE. It can also be deduced from several more basic inequalities [T1]. The proof I'll discuss today fits in this category as well. I shall derive a strong isoperimetric inequality (SI) as a consequence of a more powerful geometric inequality (B), discovered by Bonnesen in 1921 [Bo]. In most proofs of the isoperimetric inequality, the second statement is harder to prove than the first, however, Bonnesen's strong isoperimetric inequality proves both statements at once. My lecture follows the article of Osserman [O1].

For simplicity sake, I shall indicate the complete proof of the isoperimetric inequality only for convex curves. Let  $\gamma$  be a simple closed Jordan curve in the plane. To be a Jordan curve means that  $\gamma$  is the image of a function  $h : \mathbf{S}^1 \rightarrow \mathbf{E}^2$ , where  $\mathbf{S}^1$  is the standard circle and  $h$  is a homeomorphism between the topology of  $\mathbf{S}^1$  and the induced topology. The basic fact is the Jordan Curve Theorem, which says that  $\mathbf{E}^2 - \gamma$  has exactly two components, each of which has  $\gamma$  as its complete boundary. Only one of the components of  $\mathbf{E}^2 - \gamma$  is bounded. Let  $K$  denote this region along with its enclosing boundary  $\gamma$ . Let  $A$  denote the area of  $K$ , and  $L$  the length of  $\gamma$ . As  $K$  is a compact region bounded by a continuous curve,  $A$  is always finite. However, for general continuous curves the length may be infinite (so (I) is trivial). We shall therefore confine our discussion to rectifiable curves, for which the length can be defined as the supremum of lengths of polygonal paths approximating  $\gamma$  [M]. For example, piecewise  $C^1$  curves are rectifiable.

A set  $K \subset \mathbf{E}^2$  is *convex* if for every pair of points  $x, y \in K$ , the straight line segment  $\overline{xy}$  from  $x$  to  $y$  is also in  $K$ , i.e.,  $\overline{xy} \subset K$ . The bounding curve of a convex set is automatically rectifiable [G]. The *convex*

*hull* of  $K$ , denoted  $\hat{K}$ , is the smallest convex set that contains  $K$ . This is equivalent to the intersection of all halfspaces that contain  $K$ ,

$$\hat{K} = \bigcap_{\substack{C \text{ is convex} \\ C \supset K}} C = \bigcap_{\substack{H \text{ is a halfspace} \\ H \supset K}} H.$$

A halfspace is a set of the form  $H = \{(x, y) \in \mathbf{E}^2 : ax + by \leq c\}$ , where  $(a, b)$  is a unit vector and  $c$  is any real number.

A proof of the isoperimetric inequality for convex  $\hat{K}$  implies it for any set. In the first place, since  $K \subset \hat{K}$  by its definition, we have  $\hat{A} \geq A$ . On the other hand, taking convex hull reduces the boundary length because the interior segments of the boundary curve, that is the components of  $\gamma - \partial\hat{K}$  of  $\gamma$  are replaced by straight line segments in  $\partial\hat{K}$ .

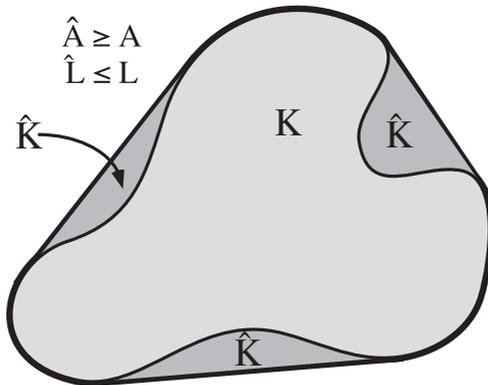


Fig. 1. The region  $K$  and its convex hull  $\hat{K}$ .

Thus, knowing the isoperimetric inequality for convex sets, we deduce

$$4\pi A \leq 4\pi \hat{A} \leq \hat{L}^2 \leq L^2.$$

Furthermore, one may also argue that if  $4\pi A = L^2$  implies  $4\pi \hat{A} = \hat{L}^2$  so that if the second part of the isoperimetric inequality holds and  $\hat{K}$  is a circle, then so is  $K$ . The basic idea is to consider the extreme points  $\partial^*\hat{K} \subset \partial\hat{K}$  of  $\hat{K}$ , that is points  $x \in \partial\hat{K}$  such that if  $x = \lambda y + (1 - \lambda)z$  for some  $y, z \in \hat{K}$  and  $0 < \lambda < 1$  then  $y = z = x$ .  $\hat{K}$  is the convex hull of its extreme points. However, the extreme points of the convex hull lie in the curve  $\partial^*\hat{K} \subset \gamma \cap \partial\hat{K}$ .  $\hat{K}$  being a circle implies that every boundary point is an extreme point, and since they come from  $\gamma$ , it means that  $\gamma$  is a circle. [R]

Let  $K$  be the region bounded by  $\gamma$ . The radius of the smallest circular disk containing  $K$  is called the circumradius, denoted  $R_{\text{out}}$ . The radius of the largest circular disk contained in  $K$  is the inradius.

$$R_{\text{in}} = \sup\{r : \text{there is } p \in \mathbf{E}^2 \text{ such that } B_r(p) \subset K\}$$

$$R_{\text{out}} = \inf\{r : \text{there exists } p \in \mathbf{E}^2 \text{ such that } K \subset B_r(p)\}$$

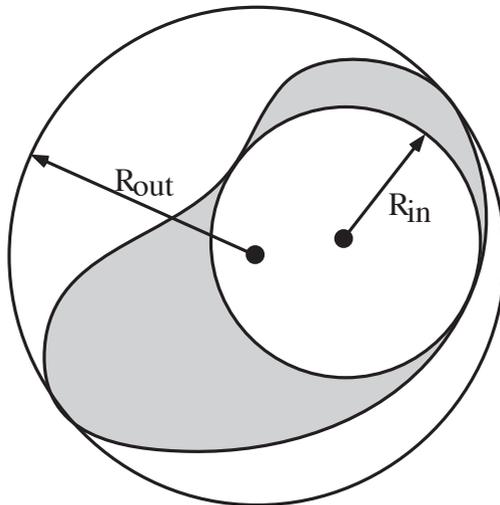


Fig. 2. The disks realizing the circumradius,  $R_{\text{out}}$ , and inradius,  $R_{\text{in}}$ , of  $K$ .

**Bonnesen's Inequality.**

Bonnesen's inequality will relate the circumradius, inradius,  $A$  and  $L$ . One of the complications in proving Bonnesen's inequality for non-convex sets by using the convex hull is that unlike the circumradius, which is the same for the convex hull and for the original domain, the inradius of the convex hull may be larger than that of the original domain. Nevertheless, Bonnesen's inequality holds for arbitrary domains.

**Bonnesen's Inequality.** *Let  $\gamma$  be a rectifiable simple closed curve in the plane of length  $L$  and which encloses an area  $A$ . Let  $R_{\text{in}}$  and  $R_{\text{out}}$  denote the inradius and circumradius of the region  $K$  enclosed by  $\gamma$ . Then*

$$(B) \quad sL \geq A + \pi s^2 \quad \text{for all } R_{\text{in}} \leq s \leq R_{\text{out}}.$$

The immediate corollary is Bonnesen's strong isoperimetric inequality.

**Corollary. Strong Isoperimetric inequality.** *Let  $\gamma$  be a rectifiable simple closed curve in the plane of length  $L$  and which encloses an area  $A$ . Let  $R_{\text{in}}$  and  $R_{\text{out}}$  denote the inradius and circumradius of the region  $K$  enclosed by  $\gamma$ . Then*

$$(SI) \quad L^2 - 4\pi A \geq \pi^2(R_{\text{out}} - R_{\text{in}})^2.$$

*Proof of corollary.* Consider the quadratic function  $f(s) = \pi s^2 - Ls + A$ . By Bonnesen's inequality,  $f(R_{\text{in}})$  and  $f(R_{\text{out}})$  are nonpositive. Hence these numbers are located between the zeros of  $f(s)$ , namely

$$R_{\text{out}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$$

$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq R_{\text{in}}.$$

Subtracting these inequalities gives

$$R_{\text{out}} - R_{\text{in}} \leq \frac{\sqrt{L^2 - 4\pi A}}{\pi},$$

which is (SI).

The strong isoperimetric inequality (SI) implies part one of the isoperimetric inequality (I), since  $\pi^2(R_{\text{out}} - R_{\text{in}})^2 \geq 0$ . Moreover, if equality holds in (I), then  $L^2 - 4\pi A = 0$  which implies that  $R_{\text{in}} = R_{\text{out}}$ , or  $\gamma$  is a circle.

*Proof of Bonnesen's inequality.* We present the proof of Fejes-Tóth. For simplicity, we shall restrict our arguments to the case that  $\gamma$  is convex. The argument of Fejes-Tóth works in general [FT]. It suffices to prove the inequality (B) for convex polygons. Intuitively, we approximate  $\gamma$  by finer and finer polygonal curves.

**Lemma 1.** *Let  $K$  be a domain bounded by a rectifiable Jordan curve with  $L$ ,  $A$ ,  $R_{\text{out}}$  and  $R_{\text{in}}$  its length, area, circumradius and inradius. Then there exists a sequence of polygons  $P_n$  that*

$$A_n \rightarrow A, \quad L_n \rightarrow L, \quad \text{and} \quad (R_n)_{\text{out}} \rightarrow R_{\text{out}}.$$

*If in addition  $K$  is convex, we may also arrange that  $(R_n)_{\text{in}} \rightarrow R_{\text{in}}$ .*

See, for example Bonnesen-Fenchel [BF] for the proof, or [O]. The rectifiability ensures first that the polygons that connect finely spaced points running along the curve have lengths that approximate the length  $L$ . By removing loops we arrange that the  $P_n$  are Jordan polygons that approximate  $\gamma$  in an appropriate manner (in Hausdorff distance). Second rectifiability ensures that  $\gamma$  itself has zero area, so that the area is approximated as well. (B) for general convex curves follows from this Lemma assuming that the inequality (B) holds for polygons by applying it to the approximations and passing to the limit.

Fejes-Tóth's proof depends on estimating the area of a parallel set of a polygon in two ways. For  $t > 0$ , the parallel set is

$$(3) \quad D_t = \{x \in \mathbf{E}^2 : d(x, P) < t\},$$

the set of points that are a distance  $t$  from  $P$ .

**Lemma 2. [Steiner's formulas].** Let  $P$  be a convex polygon,  $K$  the region enclosed by  $P$  and  $t > 0$ . Then

$$(4) \quad L(\partial(D_t \cup K)) = L + 2\pi t,$$

$$(5) \quad \text{Area}(D_t \cup K) = A + Lt + \pi t^2.$$

The result holds for convex curves also. It continues to hold for nonconvex curves except that (5) becomes an inequality " $\leq$ ."

*Proof.* The proof is elementary geometry. For each segment of  $P$  of length  $\ell$ , glue an  $\ell \times t$  rectangle to the exterior of  $P$ . The remainder of  $D_t \cup K$  consists of sectors of circles of radius  $t$  which add up to exactly one complete circle. The total length of  $\partial(K \cup D_t)$  is the sum of lengths of the segments plus the lengths of the arcs (4). The total area is the area of  $K$  plus the areas of the rectangles plus the areas of the sectors (5).

The reason Steiner's formulas fail for nonconvex curves is that the parallel set has an extra overlap. However,  $\text{Area}(D_t \cup K) \leq A + Lt + \pi t^2$  still holds.

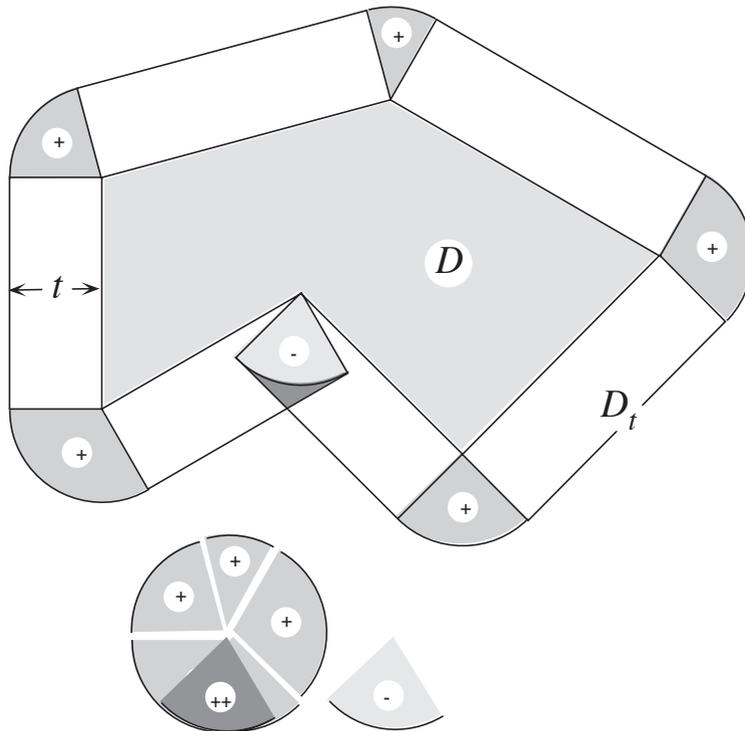


Fig. 3. The parallel set of a nonconvex polygon and Steiner's Inequality.

**Lemma 3.** Let  $P$  denote a finite polygon in the plane. Let  $D_t$  denote the set of points that are a distance less than  $t > 0$  from  $P$  as in (3). For each  $k \in \mathbf{Z}^+$ , let

$$e_k = \{x \in D_t : \#(C_t(x) \cap P) = k\}$$

denote the subset of points such that the circle  $C_t(x) = \{y \in \mathbf{E}^2 : d(x, y) = t\}$  of radius  $t$  about  $x$  meets  $P$  in  $k$  points. Then

$$(6) \quad 4tL = \sum_{k=1}^{\infty} k|e_k|.$$

In fact  $P$  can be any union of finitely many segments in this lemma.

*Proof.* The set of centers such that the circle is tangent to one of the edges of  $P$  consists of line segments parallel to the edge. The set of centers whose circles pass through the vertices consists of circular arcs about the vertices. Both of these sets have zero area, so that these sets can be ignored.

The proof is by induction on the number of edges. For a polygon of one edge,  $P_1$ , the length of the edge is  $L$  which may be more or less than  $t$ . Letting  $\Delta_1$  and  $\Delta_2$  be the open  $t$ -disks centered at the ends of the segment, the set

$$\begin{aligned} e_1 &= (\Delta_1 \cup \Delta_2) - (\Delta_1 \cap \Delta_2) \\ e_2 &= D_t - (\Delta_1 \cup \Delta_2) \end{aligned}$$

In either case, the sum of areas is  $|e_1| + 2|e_2|$  equal to the sum of the areas of the left and right crescent as in Fig. 5. Each crescent is bounded by two semicircles of radius  $t$  that are  $L$  apart, which has area  $2tL$  each.

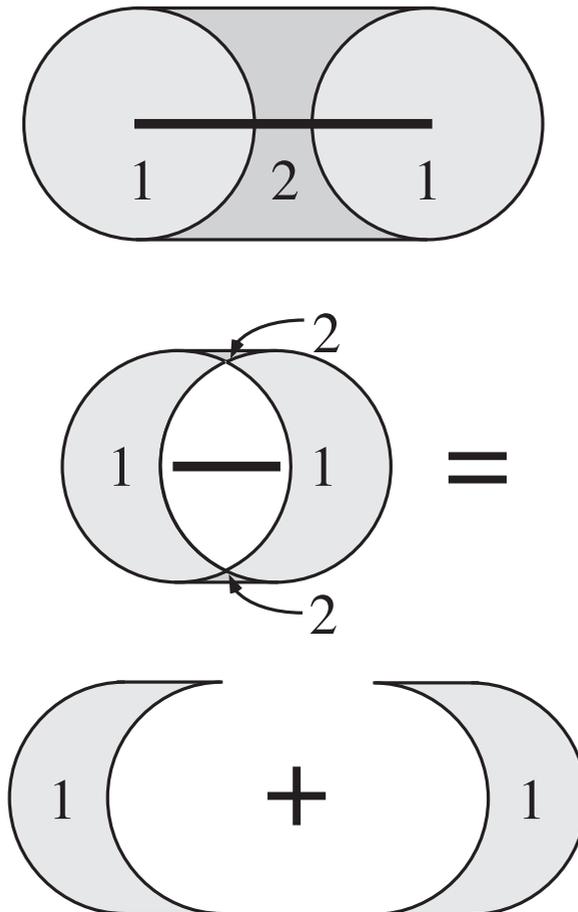


Fig. 5. Fejes-Toth's sum of areas for the parallel set of a single segment.

Now let us suppose that formula (6) holds for polygons  $P_n$  with  $n$  edges. Consider adding a new edge  $P'$ . Then the  $t$ -parallel set of the new polygon is the union of the parallel sets  $D_{n+1} = D_n \cup D'$ . Consider the sets

$$\begin{aligned} \varepsilon_j &= \{x \in D_{n+1} : \#(C_t(x) \cap P_n) = j\} \\ \varepsilon'_k &= \{x \in D_{n+1} : \#(C_t(x) \cap P') = k\} \\ \varepsilon''_\ell &= \{x \in D_{n+1} : \#(C_t(x) \cap P_{n+1}) = \ell\} \end{aligned}$$

defined for  $j, k, \ell = 0, 1, 2, 3, \dots$ . Then  $\{\varepsilon'_k\}_{k \in \mathbf{Z}_+}$  and  $\{\varepsilon_j\}_{j \in \mathbf{Z}_+}$  are partitions of  $D_{n+1}$  into disjoint sets. The lemma follows from the additive nature of area (see Fig. 6. which shows areas of parallel sets of a segment and a polygon before and after combining.)

Then we just use the fact that  $\varepsilon'_k = \coprod_j (\varepsilon'_k \cap \varepsilon_j)$  and  $\varepsilon_j = \coprod_k (\varepsilon'_k \cap \varepsilon_j)$  are disjoint unions.

$$\begin{aligned}
 L(P_{n+1}) &= L(P') + L(P_n) \\
 &= \sum_k k|\varepsilon'_k| + \sum_j j|\varepsilon_j| \\
 &= \sum_k \sum_j k|\varepsilon'_k \cap \varepsilon_j| + \sum_k \sum_j j|\varepsilon'_k \cap \varepsilon_j| \\
 &= \sum_k \sum_j (k+j)|\varepsilon'_k \cap \varepsilon_j| \\
 &= \sum_\ell \sum_{k+j=\ell} \ell|\varepsilon'_k \cap \varepsilon_j| \\
 &= \sum_\ell \ell|\varepsilon''_\ell|.
 \end{aligned}$$

We have used the observation that

$$\varepsilon''_\ell = \coprod_{k+j=\ell} (\varepsilon'_k \cap \varepsilon_j),$$

that is, the number of points of intersection of the circle with center in an  $\varepsilon_k \cap \varepsilon'_j$  is the sum of the number of points touching  $P_n$  and the number touching  $P'$ . Thus the induction step is complete and Lemma 3 is proved.

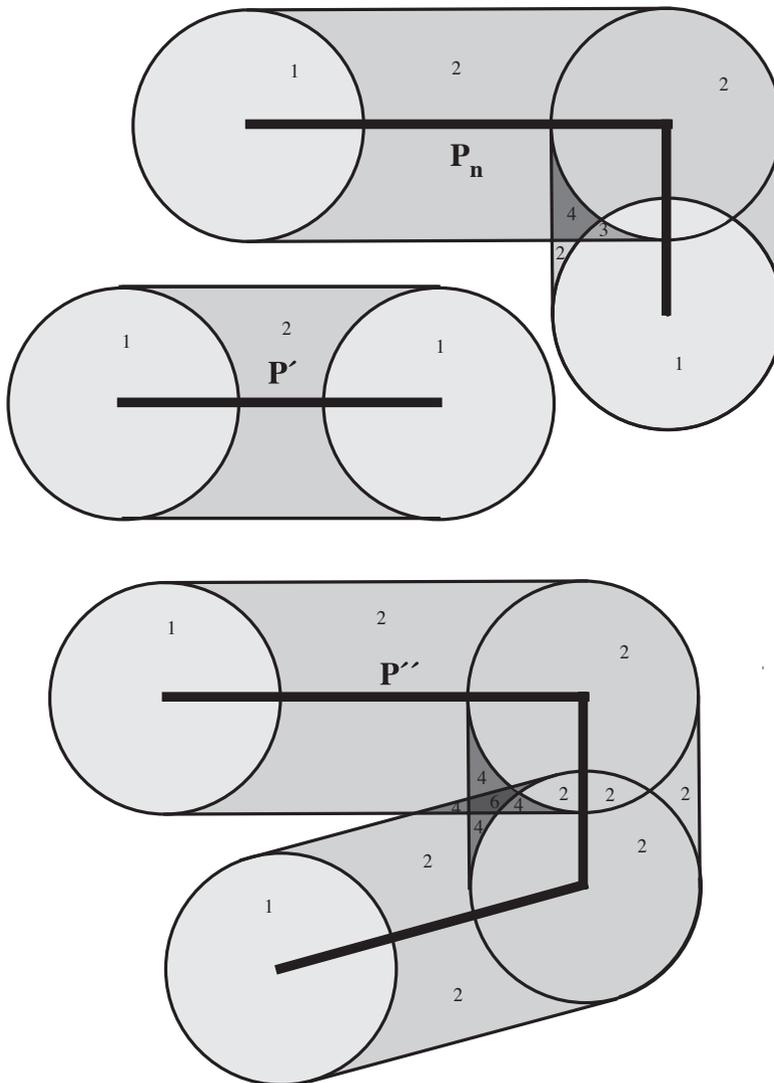


Fig. 6. Induction step: add areas of the parallel set of segment  $P'$  to the parallel set of polygon  $P_n$ .

*Proof of Bonnesen's inequality.* We shall describe Fejes-Tóth's proof for convex curves for simplicity sake. Since by Lemma 1 we may approximate convex curves, it suffices to prove (B) for convex polygons. Let  $P$  be a closed convex polygon. The idea of the proof is to estimate the area of the  $t$ -parallel set  $D_t$  in two ways.

Since  $P$  is closed, except for a set of centers of measure zero, each circle meets  $P$  at an even number of points. Thus, by Lemma 3 we have

$$\begin{aligned} 4tL &= 2|e_2| + 4|e_4| + 6|e_6| + 8|e_8| + \cdots \\ &\geq 2|e_2| + 2|e_4| + 2|e_6| + 2|e_8| + \cdots \end{aligned}$$

Let us first consider the case that  $R_{\text{in}} < t < R_{\text{out}}$ . Then the  $t$ -circle about every point of  $D_t$  must meet  $P$  because the circle is neither big enough to be contained in  $P$  nor big enough to contain  $P$ . Hence  $D_t$  contains the region bounded by  $P$ . But  $D_t = \coprod_{k \in \mathbf{N}} e_k$  ( $= \coprod_{k \in \mathbf{N}} e_{2k}$  up to a set of measure zero) so

$$\begin{aligned} 4tL &\geq 2(|e_2| + |e_4| + |e_6| + |e_8| + \cdots) \\ &= 2 \text{Area}(D_t). \end{aligned}$$

Finally, by Steiner's Formula (5),

$$(7) \quad \begin{aligned} 2tL &\geq \text{Area}(D_t) \\ &= A + Lt + \pi t^2, \end{aligned}$$

which is (B). Finally, choosing a sequence  $t_i \in (R_{\text{in}}, R_{\text{out}})$  tending to either  $R_{\text{in}}$  or  $R_{\text{out}}$  as  $i \rightarrow \infty$ , the inequality (B) can be passed to the limit, establishing (B) for all  $t \in [R_{\text{in}}, R_{\text{out}}]$ .

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