Math 5410 § 1.	**Final Exam**	Name:________________
Treibergs		December 11, 2020

1.	/12
2.	/12
3.	/12
4.	/12
5.	/12
6.	/12
7.	/12
8.	/20
Total	/90

This is an open book test. You may use your text and notes. Do not use a calculator, consult another person or use the internet. Write clearly, justify your answers and show your work to receive credit. PDF answer files must be upload to canvas within the time provided. There are [58-62] total points. **DO ONLY SEVEN PROBLEMS**. If you do more than seven problems, only the first four will be graded. Not all problems have the same point value.

1. [12] Find the general solution $\dot{x} = Ax$. [Hint: $\lambda = -2, -2, -1$]

$$A = \begin{pmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

2. [12] Determine the local stable and unstable curves at the origin for both the linearized as well as the nonlinear equations. Sketch and compare the phase portraits of the linearized equations about the origin and the nonlinear equations.

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y + x^2 \end{cases}$$

3. Consider the initial value problem

$$x' = 2tx; \qquad x(0) = x_0. \tag{IVP}$$

(a) [4] Find the integral equation (IE) satisfied by solutions of the (IVP). Give the recursion formula for the Picard Iterates $u_k(t)$ for the integral equation.

(b) [4] Assuming that the Picard Iterates converge uniformly $u_k(t) \to x(t)$ on the interval $[-a, a]$, explain why $x(t)$ satisfies both (IE) and (IVP).

(c) [4] Perform at least two iterations. Determine the limit of your Picard iterates and check that it satisfies the (IVP).

4. Consider the system
$$\begin{cases} \dot{x} = y - x \\ \dot{y} = -y - xz \\ \dot{z} = xy - z \end{cases}$$

   (a) [6] Show that there is a trapping region T such that for all starting points $c \in \mathbf{R}^3$, the trajectory stating from c will enter T.

   (b) [6] Show that a unique $\mathcal{C}^1$ solution $\varphi(t, c)$ with initial value c exists for all $t \geq 0$ and $c \in \mathbf{R}^3$. State completely any theorems that you quote, don't just give its page or section number.

5. [12] Show that the curve $C = \left\{(x,y) : \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{y^4}{4} = \dfrac{1}{2}\right\}$ is a periodic trajectory. Find the Poincaré Map for C. Determine whether C is stable and explain what sense of stability you mean. [Hint: first show that this is a Hamiltonian System.]

$$\begin{cases} \dot{x} = y + y^3 \\ \dot{y} = -x \end{cases}$$

6. [12] The system undergoes a bifurcation at the origin as the parameter a varies. Find the bifurcation values and sketch the trajectories for a above and below the bifurcation values. Identify stable and unstable trajectories. What kind of bifurcation is it? Why?

$$\begin{cases} \dot{x} = ax + y \\ \dot{y} = -x + ay - x^2 y \end{cases}$$

7. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [4] STATEMENT: The set of hyperbolic matrices A is open and dense in the set of real matrices.

TRUE: ◯	FALSE: ◯

   (b) [4] STATEMENT: Let $X(t)$ and $Y(t)$ be solutions of $\dot{Z} = F(Z)$ defined for $a < t < b$ where $F : \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable. If $X(t_0) = Y(t_0)$ for some $t_0 \in (a, b)$ then $X(t) = Y(t)$ for all $t \in (a, b)$.

TRUE: ◯	FALSE: ◯

   (c) [4] STATEMENT: If $F : \mathbf{R}^2 \to \mathbf{R}^2$ is a continuously differentiable, then the omega limit set $\omega(X_0)$ is either an equilibrium point or a closed trajectory of period $p > 0$.

TRUE: ◯	FALSE: ◯

8. Consider the predator/prey system for populations $x, y \geq 0$,

$$\begin{cases} x' = x(x(4-x) - y) \\ y' = y(x-1) \end{cases}$$

(a) [5] Find the null-clines and equilibrium points.

(b) [5] For each equilibrium point, determine the local behavior.

(c) [5] Show that there is a nontrivial periodic orbit.

(d) [5] Sketch the global flow pattern. Be sure to indicate directions of flow in each region, stable and unstable directions at the saddles and any limit cycles.