Math 5410 § 1. **First Midterm Exam** Name: Solutions
Treibergs Sept. 20, 2017

1. *Consider the system*

$$X' = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} X.$$

Sketch the regions in the ab-plane where this system has different types of canonical forms. In the interior of each region, sketch a small phase plane indicating how the flow looks.

Find the eigenvalues.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ a & b-\lambda \end{vmatrix} = \lambda^2 - b\lambda - a$$

Solving the quadratic equation

$$\lambda = \frac{b \pm \sqrt{b^2 + 4a}}{2}.$$

Thus the ab-plane is split into five regions by the parabola $4a = -b^2$ and the coordinate axes. Note that $\lambda_1\lambda_2 = \det(A) = -a$ and $\lambda_1 + \lambda_2 = \operatorname{tr} A = b$. Hence if $a > 0$ the determinant is negative and the eigenvalues have opposite signs: the rest point is a saddle. If $4a < -b^2$ then the roots are complex. If also $b < 0$ ($b > 0$) the rest point is a stable spiral (unstable spiral resp.) But if $-b^2 < 4a < 0$ the roots are real. If also $b < 0$ ($b > 0$) the rest point is a stable node (unstable node resp.)

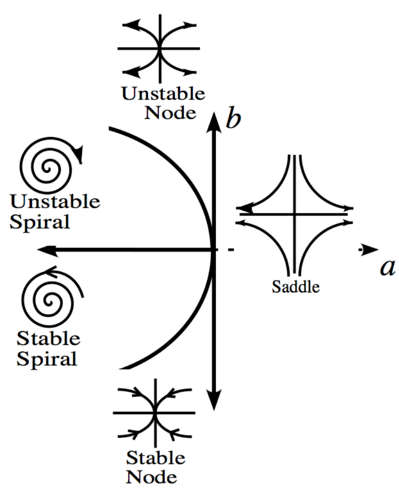

Figure 1: ab plane for Problem 1.

2. *Consider the system*

$$X' = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} X. \tag{1}$$

Find the real general solution. Determine the real canonical form $Y' = BY$ for system (1). Find the matrix M so that $Y = MX$ puts (1) in canonical form. Check that your matrix works.

Find the eigenvalues.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$$

so $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Solving for the λ_1 eigenvector

$$0 = (A - \lambda_1 I)v_1 = \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

Thus a complex solution is given by

$$\begin{aligned} X(t) = e^{(2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix} &= e^{2t}(\cos t + i \sin t) \begin{pmatrix} 2 \\ 1+i \end{pmatrix} \\ = e^{2t} &\left[\begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix} + i \begin{pmatrix} 2 \sin t \\ \cos t + \sin t \end{pmatrix} \right] \end{aligned}$$

The real general solution is a combination of the real and imaginary parts of one of the complex solutions.

$$X(t) = e^{2t} \left[c_1 \begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t \\ \cos t + \sin t \end{pmatrix} \right].$$

If $\lambda = a + ib$ then the real canonical form is $Y' = BY$ where

$$B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

There is a matrix T such that $B = T^{-1}AT$ and the transformation is given by $M = T^{-1}$. Indeed, if $Y = T^{-1}X$ then

$$Y' = T^{-1}X' = T^{-1}AX = T^{-1}ATY = BY.$$

In fact, the matrix is given by the real and imaginary parts of the eigenvector

$$T = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M = T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

To check, we compute

$$AT = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

which equals

$$TB = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

3. *Let A be an $n \times n$ real matrix. Define "*$\operatorname{range} A$*" and "*$\ker A$*." Let A be an $n \times n$ real matrix such that* $\ker A = \{0\}$*. From first principles, show that* $\operatorname{range} A = \mathbf{R}^n$ *and, therefore* $\dim \ker A + \dim \operatorname{range} A = n$*.*

The kernel is the nullspace defined by

$$\ker A = \{x \in \mathbf{R}^n : Ax = 0\}.$$

The range is the image defined by

$$\operatorname{range} A = \{Ay : y \in \mathbf{R}^n\}.$$

Suppose that the kernel is zero. That means that the only solution of

$$Ax = 0$$

is $x = 0$. If we do elementary row operations R, the matrix A is reduced to a reduced row echelon form that has no free columns, otherwise there are nonzero null vectors. But an $n \times n$ reduced row echelon matrix with no free columns is the identity matrix

$$RA = I.$$

We claim that $\operatorname{range} A = \mathbf{R}^n$. To see this, we show that any $b \in \mathbf{R}^n$ is the image of some vector x under A. Such x satisfies

$$Ax = b.$$

Doing row operations

$$x = Ix = RAx = Rb.$$

Since we found an $x \in \mathbf{R}^n$ such that $b = Ax$, any vector $b \in \mathbf{R}^n$ is in the range of A. $\square$

4. *Consider the family of differential equations depending on the parameter a.*

$$x' = x^3 + 4x^2 - ax$$

Find the bifurcation points. Sketch the phase lines for values of a just above and just below the bifurcation values. Sketch the bifurcation diagram for this family of equations. Determine the stability type of the rest points for each a.

Factoring,

$$x' = x(x^2 + 4x - a) = f(x, a).$$

The bifurcation curves are the solutions of $f(x, a) = 0$ which are the curves $x = 0$ and $a = x^2 + 4x = (x+2)^2 - 4$. Thus $x = 0$ is a rest point for all values of a and $x = -2 \pm \sqrt{a+4}$ are two more rest points for $a > -4$. Thus there are two bifurcation points at $(a, x) = (-4, -2)$ and at $(a, x) = (0, 0)$. As a increases from $-\infty$, a rest point appears at $a = -4$ which splits into a stable and unstable rest point for $-4 < a$ giving a fold type bifurcation. Then as a increases through $a = 0$, a stable and unstable rest point collide and "bounce," giving a transcritical bifurcation. The phase lines are indicated for some typical a values in Fig. 2. Since $f(x, a)$ goes from negative to positive at $x = -2 \pm \sqrt{a+4}$ when $a > 0$, these are both unstable. $x = 0$ is stable for $a > 0$ and unstable for $a < 0$. $x = -2 + \sqrt{a+4}$ is stable for $-4 < a < 0$ and $x = -2 - \sqrt{a+4}$ is unstable for $a > -4$. The flow directions are indicated on the $a =$ const. lines for some typical values of a. When $a < -4$ when $x \mapsto f(x, a)$ is an increasing function which is negative for $x < 0$ and positive for $x > 0$. Thus flow is away from the rest point. When $-4 < a < 0$, $x \mapsto f(x, a)$ goes from negative to positive to negative to positive so flow is to the left for $x < -2 - \sqrt{4+a}$ and $-2 + \sqrt{4+a} < x < 0$ and to the right otherwise making the rest points $-2 - \sqrt{4+a}$ and 0 unstable and $-2 + \sqrt{4+a}$ stable. When $0 < a$, $x \mapsto f(x, a)$ goes from negative to positive to negative to positive so flow is to the left for $x < -2 - \sqrt{4+a}$ and $0 < x < -2 + \sqrt{4+a}$ and to the right otherwise making the rest points $-2 \pm \sqrt{4+a}$ unstable and 0 stable.

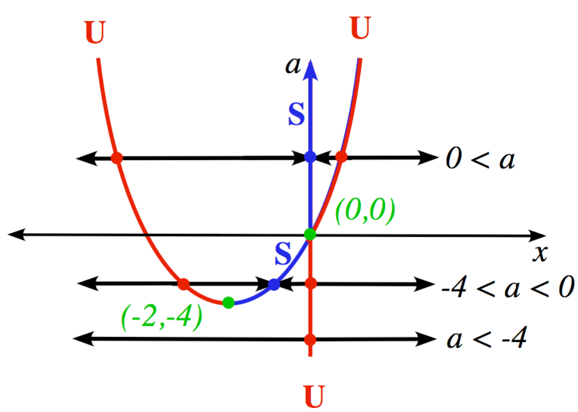

Figure 2: Bifurcation Diagram and Phase Lines for Problem (4).

5. *Find the flows* ϕ_t^X *and* ϕ_t^Y*. Find an explicit congugacy between the flows and check that your conjugacy works.*

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} X, \qquad Y' = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

If the flow starts at (a, b) at $t = 0$, the flows are given by solving the systems

$$\phi_t^X \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^t a \\ e^{-3t} b \end{pmatrix}, \qquad \phi_t^Y \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{-2t} a \\ e^{2t} b \end{pmatrix}.$$

Notice that the incoming and outgoing axes are different, so we seek a homeomorphism that swaps the two directions. We look for p and q so that

$$h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(y)|y|^p \\ \operatorname{sgn}(x)|x|^q \end{pmatrix}.$$

Then flowing first and then applying the map yields

$$h \circ \phi_t^X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(y)|e^{-3t} y|^p \\ \operatorname{sgn}(x)|e^t x|^q \end{pmatrix}.$$

Applying the map first and then flowing yields

$$\phi_t^Y \circ h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-2t} \operatorname{sgn}(y)|y|^p \\ e^{2t} \operatorname{sgn}(x)|x|^q \end{pmatrix}.$$

For these to be equal we need

$$3p = 2, \qquad 2 = q \qquad \Longrightarrow \qquad p = \frac{2}{3}, \qquad q = 2$$

so

$$h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(y)|y|^{2/3} \\ \operatorname{sgn}(x)|x|^2 \end{pmatrix}.$$

Checking,

$$h \circ \phi_t^X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(y)|e^{-3t} y|^{2/3} \\ \operatorname{sgn}(x)|e^t x|^2 \end{pmatrix} = \begin{pmatrix} e^{-2t} \operatorname{sgn}(y)|y|^{2/3} \\ e^{2t} \operatorname{sgn}(x)|x|^2 \end{pmatrix} = \phi_t^Y \circ h \begin{pmatrix} x \\ y \end{pmatrix}.$$