Math 3210 § 3. **First Midterm Exam** Name: Solutions
Treibergs September 10, 2014

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1. *Show that $2^n + 3^n$ is a multiple of 5 for all odd n in* $\mathbf{N}$.

   Odd numbers are given by $2n-1$ as n runs through $\mathbf{N}$. The statements being proved are

   $$\mathcal{P}_n = \text{“ } 2^{2n-1} + 3^{2n-1} \text{ is a multiple of 5, ”}$$

   where $n \in \mathbf{N}$. We argue by induction. For the base case $n = 1$, the statement $\mathcal{P}_1$ is "$2^1 + 3^1 = 5 \cdot 1$ is a multiple of 5" which is true.

   For the induction case, assume that for some $n \in \mathbf{N}$ that $2^{2n-1} + 3^{2n-1}$ is a multiple of 5. Now for $n+1$,

   $$2^{2(n+1)-1} + 3^{2(n+1)-1} = 4 \cdot 2^{2n-1} + 9 \cdot 3^{2n-1} = 4 \cdot (2^{2n-1} + 3^{2n-1}) + 5 \cdot 3^{2n-1}.$$

   By the induction hypothesis the first summand is a multiple of 5 and the second summand has 5 as a factor. Since both are multiples of 5 it follows that $2^{2(n+1)-1} + 3^{2(n+1)-1}$ is a multiple of 5, which is $\mathcal{P}_{n+1}$.

   Since both cases hold, by induction, for all $n \in \mathbf{N}$, $2^{2n-1} + 3^{2n-1}$ is a multiple of 5.

2. Recall the axioms of a field $(F, +, \times)$. For any $x, y, z \in F$,

   A1. (Commutativity of Addition.) $x + y = y + x$.

   A2. (Associativity of Addition.) $x + (y + z) = (x + y) + z$.

   A3. (Additive Identity.) $(\exists\, 0 \in F)\,(\forall\, t \in F)\ 0 + t = t$.

   A4. (Additive Inverse) $(\exists\, -x \in F)\ x + (-x) = 0$.

   M1. (Commutativity of Multiplication.) $xy = yx$.

   M2. (Associativity of Multiplication.) $x(yz) = (xy)z$.

   M3. (Multiplicative Identity.) $(\exists\, 1 \in F)\ 1 \neq 0$ and $(\forall\, t \in F)\ 1t = t$.

   M4. (Multiplicative Inverse.) If $x \neq 0$ then $(\exists\, x^{-1} \in F)\ x^{-1}x = 1$.

   D. (Distributivity) $x(y + z) = xy + xz$.

   Using only the axioms of a field, show that the multiplicative identity is unique. Justify every step of your argument using just the axioms listed here.

   Assume a and b are multiplicative identities. We wish to show that $a = b$ so that all multiplicative identities are the same and are called "1."

   Since we assume a is a multiplicative identity, by M3, $a \neq 0$ and $(\forall t \in F)\, at = t$. In particular, for $t = b$ we have $ab = b$.

   Since we assume b is a multiplicative identity, by M3, $b \neq 0$ and $(\forall t \in F)\, bt = t$. In particular, for $t = a$ we have $ba = a$.

   By commutativity of multiplication M1, $a = ba = ab = b$, as to be shown.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Let* $f : \mathbf{A} \to \mathbf{B}$ *be a function. If for all* $x \in \mathbf{A}$ *there is a* $y \in \mathbf{B}$ *such that* $f(x) = y$ *then* f *is onto.*

   FALSE. The statement is true of any function. *eg.*, if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = 0$ then for every x there is a y, namely $y = 0$ so that $f(x) = y$. But this f is not onto since $y = 1$ is not an image point.

   (b) Statement: *Let* $f : A \to B$ *and* $E \subset A$ *be a subset. Then* $E = f^{-1}\big(f(E)\big)$.

   FALSE. Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$ and let $E = [1, 2]$. Then $f(E) = [1, 4]$ and $f^{-1}(f(E)) = [-2, -1] \cup [1, 2] \neq E$.

   (c) Statement: *Suppose* $E, G \subset A$. *If* $f : A \to B$ *is a one-to-one function then* $f(E) = f(G)$ *implies* $E = G$.

   TRUE. We show if $x \in E$ then $x \in G$ and if $x \in G$ then $x \in E$. To show the first claim, if $x \in E$ then $f(x) \in f(E) = f(G)$ so there is $z \in G$ so that $f(z) = f(x)$. Since f is one-to-one, we have $x = z$ so $x \in G$. The second claim is symmetric with the roles of E and G swapped.

4. Let $(F, +, \times)$ be a field. F is an *ordered field* if it has a relation "$\leq$" that satisfies these additional axioms. For any $x, y, z \in F$,

   O1. (Comparability Property.) $x \leq y$ or $y \leq x$.

   O2. (Trichotomy Property.) If $x \leq y$ and $y \leq x$ then $x = y$.

   O3. (Transitivity Property.) If $x \leq y$ and $y \leq z$ then $x \leq z$.

   O4. (Additivity Property.) If $x \leq y$ and then $x + z \leq y + z$.

   O5. (Multiplicative Property.) If $x \leq y$ and $0 \leq z$ then $xz \leq yz$.

Show that $0 \leq a \leq b$ *implies* $a^2 \leq b^2$. *Justify every step of your argument using just the field axioms and axioms listed here.*

(a) Assumptions $0 \leq a$ and $a \leq b$ imply $a^2 \leq ba$ using the multiplicative property O5.

(b) Assumptions $0 \leq a$ and $a \leq b$ imply $0 \leq b$ using transitivity O3.

(c) The result from (b) $0 \leq b$ and the assumption $a \leq b$ imply $ab \leq b^2$ using the multiplicative property O5.

(d) The result from (c) $ab \leq b^2$ implies $ba \leq b^2$ using the commutative property of multiplication M1.

(e) The results from (a) $a^2 \leq ba$ and from (d) $ba \leq b^2$ imply $a^2 \leq b^2$ using the transitive property O3. □

5. *Let* $E \subset \mathbb{R}$ *be a set of real numbers given by*

$$E = \{x \in \mathbf{R} : (\exists \sigma > 0) \quad (\forall \tau > \sigma) \quad (\sigma \le x \le \tau) \quad \}.$$

Find a simple expression for E *in terms of intervals and prove your result.*

The set may be written

$$E = \bigcup_{\sigma > 0} \bigcap_{\tau > \sigma} [\sigma, \tau] = \bigcup_{\sigma > 0} \{\sigma\} = (0, \infty).$$

To prove it we show if $x \in E$ then $x \in (0, \infty)$ and if $x \in (0, \infty)$ then $x \in E$.

Suppose $x \in E$. Then there exists $\sigma_0 > 0$ such that $(\forall \tau > \sigma_0)(\sigma_0 \le x \le \tau)$. Hence $0 < \sigma_0 \le x$ which says $x \in (0, \infty)$.

On the other hand, if $x \in (0, \infty)$ then $0 < x$. If one takes $\sigma = x > 0$ then $(\forall \tau > \sigma)(\sigma \le x \le \tau)$ which is the condition that $x \in E$. □