Math 3210 § 2. **Sample Problems for First Midterm** Name: SAMPLE
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First Midterm Exam given Sept. 20, 2000.

1. *Using induction, prove that for all $n \in \mathbb{N}$,*

$$(\mathcal{P}(n)) \qquad 1 + 3 + \cdots + (2n-1) = n^2$$

2. *Let $f : X \to Y$ be a function. Suppose that there is a function $g : Y \to X$ so that $g \circ f$ is the identity and that $f \circ g$ is the identity. Show that f is* one-to-one *and* onto.

3. *Assuming only the field axioms for $\mathbf{R}$, deduce that for every $x \in \mathbf{R}$ there holds $x \cdot 0 = 0$. For each step of your deductions, state which axiom is being used.*

4. *Find the complement in $\mathbf{R}$ of the set of numbers $x \in \mathbf{R}$ for which there exists $\varepsilon > 0$ such that $x \le -\varepsilon$ or $x \ge \varepsilon$. Written in symbols $\forall$, $\exists$, $\backslash$, you are to find the set*

$$E = \mathbf{R} \backslash \{x \in \mathbf{R} : (\exists \varepsilon > 0)(x \le -\varepsilon \text{ OR } \varepsilon \le x)\}.$$

5. *Using Peano's axioms and their immediate consequences proved in class, show that if $m, n \in \mathbb{N}$ then $m + n \ne n$. [Hint: use induction on n.]*

Extra Problems.

E1. *The terms of a sequence $a_0, a_1, a_2, a_3, \ldots$ are given by $a_0 = 0$, $a_1 = 1$ and the recursive relation for $n \ge 1$ by $a_{n+1} = 2a_n - a_{n-1} + 2$. Find a formula for a_n and prove it.*

E2. *For $x, y \in \mathbf{R}$, say that x and y satisfy the relation $P(x, y)$ whenever $x = y + i$ for some $i \in \mathbf{Z}$. Show that P is an equivalence relation. Describe $\mathbf{R}/P$.*

E3. *Let $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}$ such that $n \ne 0\}/\sim$ be the usual definition of the rational numbers, where we declare two fractions equivalent, $\frac{m}{n} \sim \frac{a}{b}$, whenever $mb = na$. Show that the usual rule for multiplication of equivalence classes $\left[\frac{m}{n}\right] \cdot \left[\frac{a}{b}\right] := \left[\frac{ma}{nb}\right]$ is well defined.*

Solutions.

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1. *Prove that for all $n \in \mathbb{N}$,*

$(A_n) \qquad 1 + 3 + \cdots + (2n-1) = n^2.$

Induction proofs have two steps: the basis step proving A_1 and the induction step $A_n \implies A_{n+1}$. First we show the basis step A_1. When $n = 1$, $1 + 3 + \cdots + (2n-1) = 1$ and $n^2 = 1$ which are equal, so A_1 is true.

Then we show the induction step. We assume the induction hypothesis: for any n we have A_n is true, namely, $1 + 3 + \cdots + (2n-1) = n^2$. We wish to show this implies A_{n+1}, namely, $1 + 3 + \cdots + (2n-1) + \big(2(n+1) - 1\big) = (n+1)^2$. However, using the induction hypothesis on the first n terms, and then rearranging,

$$\big\{1 + 3 + \cdots + (2n-1)\big\} + \big[2(n+1) - 1\big] = \big\{n^2\big\} + \big[2n + 1\big] = (n+1)^2,$$

so the induction step is complete.

As the basis and the induction steps hold, by induction, A_n holds for all n.

2. *Let $f : X \to Y$ be a function. Suppose that there is a function $g : Y \to X$ so that $g \circ f$ is the identity and that $f \circ g$ is the identity. Show that f is* one-to-one *and* onto.

First we show that f is onto, namely, for every $y \in Y$ there is an $x \in X$ so that $y = f(x)$. Choose $y \in Y$. The desired x is $x = g(y)$. To see that this x works, $f(x) = f(g(y)) = (f \circ g)(y) = \text{Id}(y) = y$ since $f \circ g = \text{Id}$. Hence we have shown f is onto.

Second we show that f is one-to-one, namely, if whenever for some $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$, then $x_1 = x_2$. Suppose there are $x_1, x_2 \in X$ so that $f(x_1) = f(x_2)$. Then apply g to both sides: $g(f(x_1)) = g(f(x_2))$ or $(g \circ f)(x_1) = (g \circ f)(x_2)$. But since $g \circ f = \text{Id}$, $\text{Id}(x_1) = \text{Id}(x_2)$ or $x_1 = x_2$. Thus we have shown that f is one-to-one.

3. *Assuming only the field axioms for $\mathbf{R}$, deduce that for every $x \in \mathbf{R}$ there holds $x \cdot 0 = 0$. For each step of your deductions, state which axiom is being used.*

Choose $x \in \mathbf{R}$.

$$\begin{aligned}
x\cdot 0 =& x\cdot 0+0 && \text{Property of additive identity.}\\
=& x\cdot 0+(x+(-x)) && \text{Additive inverse of } x.\\
=& (x\cdot 0+x)+(-x) && \text{Associativity of addition.}\\
=& (0\cdot x+x)+(-x) && \text{Commutativity of multiplication.}\\
=& (0\cdot x+1\cdot x)+(-x) && \text{Multiplicative identity.}\\
=& (0+1)\cdot x+(-x) && \text{Distributive. (From the right.)}\\
=& (1+0)\cdot x+(-x) && \text{Commutativity of addition.}\\
=& 1\cdot x+(-x) && \text{Property of additive identity.}\\
=& x+(-x) && \text{Multiplicative identity.}\\
=& 0 && \text{Additive inverse of } x.
\end{aligned}$$

Thus $x \cdot 0 = 0$ and we are done.

4. *Find the set* $E = \mathbf{R}\setminus\{x \in \mathbf{R} : (\exists \varepsilon > 0)(x \le -\varepsilon \text{ } OR \text{ } \varepsilon \le x)\}$.

$$\begin{aligned}
E =& \mathbf{R}\setminus\{x \in \mathbf{R} : (\exists \varepsilon > 0)(x \le -\varepsilon \text{ OR } \varepsilon \le x)\}\\
=& \{x \in \mathbf{R} :\sim (\exists \varepsilon > 0)(x \le -\varepsilon \text{ OR } \varepsilon \le x)\} && \text{Meaning of complement.}\\
=& \{x \in \mathbf{R} : (\forall \varepsilon > 0) \sim (x \le -\varepsilon \text{ OR } \varepsilon \le x)\} && \text{Negation of } \exists.\\
=& \big\{x \in \mathbf{R} : (\forall \varepsilon > 0)\big(\sim (x \le -\varepsilon) \text{ AND } \sim (\varepsilon \le x)\big)\big\} && \text{De Morgan's Law.}\\
=& \{x \in \mathbf{R} : (\forall \varepsilon > 0)(-\varepsilon < x \text{ AND } x < \varepsilon)\}\\
=& \{x \in \mathbf{R} : 0 \le x \text{ AND } x \le 0\}\\
=& \{x \in \mathbf{R} : x = 0\}\\
=& \{0\}\,.
\end{aligned}$$

5. *Using Peano's axioms and their immediate consequences proved in class, show that if* $m, n \in \mathbb{N}$ *then* $n + n \ne n$.

Choose $m \in \mathbb{N}$. Let $\mathcal{Q}(n)$ be the statement "$m + n \ne n$."

The basis statement $\mathcal{Q}(1)$ is $m+1 \ne 1$. Arguing by contradiction, if this were not the case then $m+1 = 1$ which says that 1 is the successor of m. However, by axiom **N3.**, 1 is not the successor of any element of $\mathbb{N}$, which implies the contradicrtion: 1 is not the successor of m.

The induction step is to show $\mathcal{Q}(n+1)$ assuming $\mathcal{Q}(n)$. In other words, we have to show $m+(n+1) \ne n+1$. Again, argue by contradiction and assume that $\ell = m + (n + 1) = n + 1$. The last equality says that ℓ is the successor to n. Using the inductive definition of addition ($m + (n + 1) := (m + n) + 1$, or its consequence, the associative property of addition in $\mathbb{N}$,) we see that $\ell = (m + n) + 1$. In other words, ℓ is the successor of $m + n$. By the inductive hypothesis $m + n \ne n$ so that ℓ is the successor of two different numbers, n and $m + n$. However, by Peano's axiom **N4.**, if two elements of $\mathbb{N}$ have the same successor, then they are equal. In particular, this implies the contradiction that n and $m + n$ are equal.

E1. *The terms of a sequence* $a_0, a_1, a_2, a_3, \ldots$ *are given by* $a_0 = 0$, $a_1 = 1$ *and the recursive relation for* $n \ge 1$ *by* $a_{n+1} = 2a_n - a_{n-1} + 2$. *Find a formula for* a_n *and prove it.*

Let's try a few terms to see the pattern. $a_2 = 2a_1 - a_0 + 2 = 2\cdot 1 - 0 + 2 = 4$. $a_3 = 2a_2 - a_1 + 2 = 2\cdot 4 - 1 + 2 = 9$. $a_4 = 2a_3 - a_2 + 2 = 2 \cdot 9 - 4 + 2 = 16$. It seems that $a_n = n^2$. Let's prove it by strong induction.

There are two base cases: for $n = 0$ we have $a_0 = 0 = 0^2$ and for $n = 1$ we have $a_1 = 1 = 1^2$.

For strong induction for $n \ge 1$, we shall show the statement for $n + 1$ assming it's true for n and $n - 1$. Using the recursive definition, $a_{n+1} = 2a_n - a_{n-1} + 2$. Using the two induction hypotheses, $a_n = n^2$ and $a_{n-1} = (n - 1)^2$ we see that $a_{n+1} = 2n^2 - (n - 1)^2 + 2 = 2n^2 - [n^2 - 2n + 1] + 2 = n^2 + 2n + 1 = (n + 1)^2$. The induction is proven.

E2. *For $x, y \in \mathbf{R}$, say that x and y satisfy the relation $P(x,y)$ whenever $x = y + i$ for some $i \in \mathbb{Z}$. Show that P is an equivalence relation. Describe $\mathbf{R}/P$.*

To be an equivalence relation, P has to be reflexive, symmetric and transitive. To see reflexive, choose $x \in \mathbf{R}$ to see if $P(x,x)$ holds. But by taking $0 \in \mathbb{Z}$, we see that $x = x + 0$ so $P(x,x)$ holds. To see transitive, for any $x, y \in \mathbf{R}$ to see if $P(x,y) \implies P(y,x)$. If $P(x,y)$ then $x = y + j$ for some $j \in \mathbb{Z}$. But by subtracting j we see that $y = x + (-j)$, where $-j \in \mathbb{Z}$. Hence $P(y,x)$ holds as well. Finally, for any $z, y, x \in \mathbf{R}$, transitivity means if $P(x,y)$ and $P(y,z)$ hold then $P(x,z)$ holds. But $P(x,y)$ means $x = y + i$ and $P(y,x)$ means $y = z + j$ for some $i, j \in \mathbb{Z}$. Substituting, this gives $x = (z + j) + i$ or $x = z + (i + j)$ for this $i + j \in \mathbb{Z}$. But this is the condition that $P(x,z)$ holds. $\mathbf{R}/P$ is nothing more than the circle. None of the points of the interval $[0,1)$ are identified to each other, because they don't differ by an integer. However, every real is identified to a point in $[0,1)$. Since 0 and 1 are identified ($P(0,1)$ holds since $0 = 1 + (-1)$) as if we glued the ends of the interval together. But this is a circle of unit length.

E3. *Let $\mathbb{Q} = \{\frac{m}{n} : \ m, n \in \mathbb{Z} \text{ such that } n \neq 0 \ \}/\sim$ be the usual definition of the rational numbers, where we declare two fractions equivalent, $\frac{m}{n} \sim \frac{a}{b}$, whenever $mb = na$. Show that the usual rule for multiplication of equivalence classes $\left[\frac{m}{n}\right] \cdot \left[\frac{a}{b}\right] := \left[\frac{ma}{nb}\right]$ is well defined.*

To be well defined on equivalence classes means that if we take different representatives of the equivalence classes, we still get the same answer. That is if $\left[\frac{m}{n}\right] = \left[\frac{m'}{n'}\right]$ and $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$ then $\left[\frac{ma}{nb}\right] = \left[\frac{m'a'}{n'b'}\right]$. The first equation means $mn' = m'n$ and the second $ab' = a'b$. Multiplying these equations we see that $man'b' = nbm'a'$. However this says $\frac{ma}{nb} \sim \frac{m'a'}{n'b'}$ as to be shown.