Math 3210 § 2. **Third Midterm Exam** Name:____________________

Treibergs November 11, 2009

[1.] *Using just the definition of derivative, show that* $f(x) = \dfrac{1}{\sqrt{x}}$ *is differentiable at* $c = 9$ *and find* $f'(9)$.

The derivative exists if the limit of the difference quotient exists. Since $x > 0$ as $x \to 9$, by the "workhorse theorem," the limit of the quotient of products exists and is the quotient of the products of the limits. Hence f is differentiable and its derivative is

$$\lim_{x\to 9} \frac{f(x) - f(9)}{x - 9} = \lim_{x\to 9} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{9}}}{x - 9} = \lim_{x\to 9} \frac{\sqrt{9} - \sqrt{x}}{\sqrt{x}\sqrt{9}(x-9)} = \lim_{x\to 9} \frac{(\sqrt{9} - \sqrt{x})(\sqrt{9} + \sqrt{x})}{\sqrt{x}\sqrt{9}(x-9)(\sqrt{9} + \sqrt{x})} =$$

$$\lim_{x\to 9} \frac{9 - x}{\sqrt{x}\sqrt{9}(x-9)(\sqrt{9} + \sqrt{x})} = \lim_{x\to 9} \frac{-1}{\sqrt{x}\sqrt{9}(\sqrt{9} + \sqrt{x})} = -\frac{1}{2\cdot 9^{3/2}} = -\frac{1}{54} = f'(9).$$

[2.] *State the definition:* $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy Sequence. Let* $x_1 \in \mathbb{R}$ *be any number. Define the sequence recursively by* $x_{n+1} = 3 - \frac{1}{2}x_n$ *for all* $n \in \mathbb{N}$. *Show that* $|x_{n+2} - x_{n+1}| \le \frac{1}{2}|x_{n+1} - x_n|$ *for all* $n \in \mathbb{N}$. *Using the definition, show that this* $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy Sequence.*

Definition: $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is a Cauchy Sequence if for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that $|x_k - x_\ell| < \epsilon$ whenever $k, \ell \in \mathbb{R}$ satisfy $k, \ell > N$.

Computing for each n,

$$|x_{n+2} - x_{n+1}| = \left| 3 - \frac{1}{2}x_{n+1} - \left(3 - \frac{1}{2}x_n\right)\right| = \frac{1}{2}|x_{n+1} - x_n|.$$

Thus, by induction, it follows that for each $n \in \mathbb{N}$,

$$|x_{n+1} - x_n| \le \frac{1}{2^{n-1}}|x_2 - x_1|.$$

To complete the argument that the sequence is Cauchy, choose $\epsilon > 0$. Let $N = 3 + \frac{|x_2 - x_1|}{\epsilon}$. Then, if $k, \ell \in \mathbb{N}$ satisfy $k, \ell > N$ either $k = \ell$ so $|x_k - x_\ell| = 0 < \epsilon$ or we may assume, after swapping if necessary, that $k > \ell$. Then sneaking in intermediate terms, using the inequality above and the formula for a geometric sum $1 + r + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r}$, and $2^n \ge n$,

$$|x_k - x_\ell| = |x_k - x_{k-1} + x_{k-1} - \cdots - x_\ell| \le |x_k - x_{k-1}| + |x_{k-1} - x_{k-2}| + \cdots + |x_{\ell+1} - x_\ell| \le$$

$$\left(\frac{1}{2^{k-2}} + \frac{1}{2^{k-3}} + \cdots + \frac{1}{2^{\ell-1}}\right)|x_2 - x_1| = \left(\frac{1}{2^{\ell-2}} - \frac{1}{2^{k-2}}\right)|x_2 - x_1| \le \frac{|x_2 - x_1|}{\ell - 2} \le \frac{|x_2 - x_1|}{N - 2} < \epsilon.$$

[3.] *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

1. **Statement:** *If* $f : \mathbb{R} \to \mathbb{R}$ *is not differentiable at* $c \in \mathbb{R}$ *then* f *is not continuous at* c.

   FALSE. The function $f(x) = |x|$ is not differentiable at $c = 0$ but it is continuous there.

2. **Statement:** *If* $g : [0, 1] \to \mathbb{R}$ *is a continuous function such that* $g(0) = 1$ *and* $g(1) = 0$. *then there is a point* $c \in [0, 1]$ *such that* $c = g(c)$.

   TRUE. Let $w(x) = x - g(x)$ which is a difference of continuous functions, hence continuous on $[0, 1]$. $w(0) = 0 - g(0) = -1$. $w(1) = 1 - g(1) = 1$. By the intermediate value theorem there is $c \in [0, 1]$ so that $w(c) = 0$. Thus for this c, $c = g(c)$.

3. **Statement:** *If $h : (0,1) \to \mathbb{R}$ is continuous and bounded, then it is uniformly continuous.*

   FALSE. The function $h(x) = \sin(1/x)$ is continuous and bounded on $(0,1)$ but it is not uniformly continuous. If it were uniformly continuous, it would have a continuous extension to $H : [0,1] \to \mathbb{R}$ such that $H(x) = h(x)$ for all $x \in (0,1)$. But H cannot be continuous at zero. The sequence $x_n = 1/(\pi n + \pi/2)$ tends to zero as $n \to \infty$ but $H(x_n) = (-1)^n$ does not converge to $H(0)$ as $n \to \infty$ because it doesn't even converge.

[4.] *Let $: \mathbb{R} \to \mathbb{R}$ be a function. State the definition: f is continuous at $a \in \mathbb{R}$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$, then there exist positive constants M and $\delta > 0$ so that $|f(x)| \le M$ for all $x \in \mathbb{R}$ such that $a - \delta < x < a + \delta$.*

Definition: f is continuous on $\mathbb{R}$ if for every $a \in \mathbb{R}$ and every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $x \in \mathbb{R}$ satisfies $|x - c| < \delta$.

Let $\epsilon = 1$ and apply continuity. There is a $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever the real number x satisfies $a - \delta < x < a + \delta$. For such x,

$$|f(x)| = |f(a) + f(x) - f(a)| \le |f(a)| + |f(x) - f(a)| < |f(a)| + \epsilon \le |f(a)| + 1 = M.$$

Thus we have established the desired estimate with this δ and $M = |f(a)| + 1$.

[5.] *Let $f, f_n : \mathbb{R} \to \mathbb{R}$ be functions for all $n \in \mathbb{N}$. State the definition: f_n converges pointwise to f as $n \to \infty$; State the definition: f_n converges uniformly to f as $n \to \infty$. Let $f_n(x) = \frac{1}{1+n^2+x^2}$. Does $f_n \to 0$ pointwise? Does $f_n \to 0$ uniformly? Why?*

Definition: f_n converges to f pointwise as $n \to \infty$ in $\mathbb{R}$ iff for every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$. In other words, $(\forall x \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{R})(\forall n \in \mathbb{N})(n > N \implies |f_n(x) - f(x)| < \epsilon)$.

Definition: f_n converges to f uniformly as $n \to \infty$ in $\mathbb{R}$ iff
$(\forall \varepsilon > 0)(\exists N \in \mathbb{R})(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(n > N \implies |f_n(x) - f(x)| < \epsilon)$.

To see that the convergence is pointwise, we choose $x \in \mathbb{R}$. Then, using limit theorems,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+n^2+x^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2} + 1 + \frac{x^2}{n^2}} = \frac{0}{0+1+0} = 0.$$

To see that the convergence is also uniform, choose $\epsilon > 0$. Let $N = \frac{1}{\sqrt{\epsilon}}$. Then for any choice of $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n > N$ we have

$$|f_n(x) - 0| = \left|\frac{1}{1+n^2+x^2}\right| \le \frac{1}{1+n^2} < \frac{1}{N^2} = \varepsilon.$$