Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs October 7, 2009

1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and $L \in \mathbb{R}$. State the definition: $L = \lim_{n\to\infty} a_n$. Guess the limit. Then use the definition of limit to prove that your guess is correct:* $\lim_{n\to\infty} \frac{4n+3}{2n+1}$.

$a_n \to L$ as $n \to \infty$ means that for all $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|a_n - L| < \varepsilon$ whenever $n > N$.

We show that $a_n \to L = 2$ as $n \to \infty$ Choose $\varepsilon > 0$. Let $N = \frac{1}{2\varepsilon}$. If $n > N$ then

$$|a_n - L| = \left|\frac{4n+3}{2n+1} - 2\right| = \left|\frac{(4n+3)-(4n+2)}{2n+1}\right| = \frac{1}{2n+1} < \frac{1}{2n} < \frac{1}{2N} = \frac{1}{2/(2\varepsilon)} = \varepsilon.$$

2. *State the definition: $m = \sup E$. Consider the union of intervals $E = \bigcup_{n\in\mathbb{N}} \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$. Find $\sup E$ and prove that it is the supremum.*

$m = \sup E$ means m is an upper bound for E, *i.e.,* $(\forall x \in E)(x \le m)$, and m is the least of upper bounds, *i.e.,* $(\forall \epsilon > 0)(\exists x \in E)(m - \epsilon < x)$.

For the given set, $1 = \sup E$. To see that 1 is an upper bound, choose $x \in E$. Hence $x \in \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$ for some $n \in \mathbb{N}$. For this n, $x < \frac{n+1}{n+2} < \frac{n+2}{n+2} = 1$. Thus $x \le 1$ for all $x \in E$. To see that m is least among upper bounds, for each $n \in \mathbb{N}$, let $q_n = \frac{1}{2}\left(\frac{n}{n+1} + \frac{n+1}{n+2}\right)$. Thus $q_n \in \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$ (by Problem 3a.) Hence $q_n \in E$. Now choose $\varepsilon > 0$. By the Archimedean Property, there is an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. For this n, $1 - \varepsilon < 1 - \dfrac{1}{n} < 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1} < q_n$.

Thus there is a $q_n \in E$ such that $1 - \varepsilon < q_n$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

a. **Statement.** *If $x, y \in \mathbb{R}$ are such that $x < y$ then $x < \frac{x+y}{2} < y$.*

TRUE. Adding x to both sides of $x < y$ implies $x + x < y + x$ implies $x = \frac{1}{2}(x+x) < \frac{1}{2}(x+y)$. Similarly, adding y to both sides of $x < y$ implies $x + y < y + y$ implies $\frac{1}{2}(x+y) < \frac{1}{2}(y+y) = y$.

b. **Statement.** *Let $\{a_n\}$ and $\{b_n\}$ be real, convergent sequences such that $a_n < b_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$.*

FALSE. Let $a_n = 0$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n < b_n$ for all n.
However, $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

c. **Statement.** *If $f, g : \mathbb{R} \to \mathbb{R}$ are functions that are bounded below, then* $\inf_{\mathbb{R}} f + \inf_{\mathbb{R}} g = \inf_{\mathbb{R}}(f+g)$.

FALSE. Let $f(x) = \sin x$ and $g(x) = -\sin x$. Then $\inf_{\mathbb{R}} f + \inf_{\mathbb{R}} g = (-1) + (-1) = -2$ but for all x, $f(x) + g(x) = 0$ so $\inf_{\mathbb{R}}(f+g) = 0$.

4. *Let $A = \{x \in \mathbb{R} : \ x^2 < 2x + 1\ \}$. Show that A is nonempty. Show that A is bounded above. What is the least upper bound of A? (You don't have to prove it.) Does the set A have a maximum? Why or why not?*

A is nonempty since $2 \in A$ because $4 = 2^2 < 2 \cdot 2 + 1 = 5$.

A is bounded above by, say, 3. If not, there is $x \in A$ so that $3 < x$. But then $0 < 3 - 1 < x - 1$ so $(3-1)^2 < (x-1)^2$ so $2 = (3-1)^2 - 2 < (x-1)^2 - 2 = x^2 - 2x - 1 < 0$ because $x \in A$. This is a contradiction, so 3 is an upper bound.

$m = \sup A$ is the positive root of $m^2 - 2m - 1 = 0$ which by quadratic formula is $m = \frac{2+\sqrt{2^2-4(1)(-1)}}{2} = 1 + \sqrt{2}$. For $m = \sup A$ to be its maximum, we would need that $m \in A$. But $m^2 = 2m + 1$ so $m \notin A$. Thus A does not have a maximum.

5. *Prove that if $x, y \in \mathbb{R}$ are numbers such that for all positive RATIONAL numbers $r > 0$ we have $|x - y| < r$. Then $x = y$.*

Proof by contrapositive. Suppose that $x \ne y$. Then $0 < |x-y|$. By the Archimedean Property, there is an $n \in \mathbb{N}$ so that $\frac{1}{n} < |x-y|$. Then $r = \frac{1}{n} > 0$ is a positive rational number such that $r \le |x-y|$. Thus we have established the statement $(\exists r \in \mathbb{Q})(r > 0 \text{ and } |x-y| \ge r)$. But this is the negation of the hypothesis in the theorem which is $(\forall r \in \mathbb{Q})(r > 0 \implies |x-y| < r)$.