Math 3210 § 1. **Final Exam** Name:____________________
Treibergs May 3, 2023

This is a closed book exam except that you are allowed four "cheat sheets," four 8.5"×11" pages with notes on both sides. Other notes, books, calculators, tablets, laptops, phones and text messaging devices are prohibited. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.______	/21
2.______	/21
3.______	/21
4.______	/21
5.______	/21
6.______	/22
7.______	/22
8.______	/22
9.______	/22
Total______	/150

1. (a) [3] Define what it means for $f : \mathbf{R} \to \mathbf{R}$ to be *differentiable* at the point $a \in \mathbf{R}$.

   (b) [18] Determine whether the given function is differentiable at $x = 1$ and prove your claim.

$$f(x) = \begin{cases} 2 - x^2, & \text{if } x \text{ is rational;} \\ 3 - 2x, & \text{if } x \text{ is irrational.} \end{cases}$$

2. (a) [3] Let $E \subset \mathbf{R}$. Define the *supremum* of E, $\sup E$.

   (b) [18] Consider the subset of rational numbers $E = \{q \in \mathbb{Q} : q^2 < 3\}$. Find $\sup E$ and prove your result.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] $f(x) = x^3 - 3x + b$ has at most one root in $[-1, 1]$. TRUE: ◯ FALSE: ◯

   (b) [7] There is no continuous function $f : \mathbf{R} \to \mathbb{R}$ such that for each $c \in \mathbf{R}$ the equation $f(x) = c$ has exactly two solutions. TRUE: ◯ FALSE: ◯

   (c) [7] Suppose $L = \lim_{R\to\infty} \int_R^R f(t)\,dt$ has a finite limit. Then the improper integral $\int_{-\infty}^{\infty} f(t)\,dt$ exists. TRUE: ◯ FALSE: ◯

4. (a) [10] Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous. Show that $F(x)$ is differentiable and find $F'(x)$, where $F(x) = \displaystyle\int_{x-1}^{x+1} f(t)\,dt$.

(b) [11]Assume that $f : [a, b] \to \mathbf{R}$ is continuous and $f(x) \ge 0$ for all $x \in [a, b]$. Prove that if $\displaystyle\int_a^b f(t)\,dt = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

5. Determine whether the following series converge. Why?

(a) [7] $S \sim \frac{1}{1} + \frac{1\cdot 3}{1\cdot 4} + \frac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7} + \cdots + \frac{1\cdot 3\cdot 5\cdots(2k-1)}{1\cdot 4\cdot 7\cdots(3k-1)} + \cdots$

CONVERGES: ◯ DOES NOT CONVERGE: ◯

(b) [7] $T \sim \sum_{k=1}^{\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k}$ CONVERGES: ◯ DOES NOT CONVERGE: ◯

(c) [7] $U \sim \sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{k}$ CONVERGES: ◯ DOES NOT CONVERGE: ◯

6. (a) [3] Define: the sequence $\{S_n\}$ is a *Cauchy Sequence*.

   (b) [19] Let $\{a_k\}$ be a real sequence such that the series $\sum_{k=1}^{\infty} |a_k|$ converges. Show that $\sum_{k=1}^{\infty} a_k$ also converges.

7. (a) [3] Let $I = (0,1)$ and $f, f_k : I \to \mathbf{R}$. Define: $f_k(x) \to f(x)$ *converges uniformly* on I as $k \to \infty$.

   (b) [19] Suppose that $\{f_k\}$ is a sequence of functions that converges uniformly to $f(x)$ on I and for each k, $f_k(x)$ is bounded on I. Prove that $f(x)$ is bounded on I.

8. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Complete the statement of the theorem. [Of several possible answers, select the one you prefer for part (b).]

   Theorem. *The bounded function* f *is integrable on* $[a, b]$ *if and only if*

   (b) [19] Using only the theorem in (a), show that if $f(x)$ is integrable on $[a, b]$ then the positive part f^+ is integrable on $[a, b]$, where

   $$f^+(x) = \begin{cases} f(x), & \text{if } f(x) > 0; \\ 0, & \text{if } f(x) \le 0. \end{cases}$$

9. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Define what it means for f to be *integrable* on $[a, b]$ and what the *Riemann integral* of f on $[a, b]$ is.

   (b) [19] Suppose $\int_a^b f(x)\,dx$ exists and is positive. Prove that there exists an interval $J \subset [a, b]$ and a constant $m > 0$ such that $f(x) \ge m$ for all $x \in J$.
   $\left[\text{Hint: consider } \underline{\int_a^b} f(x)\,dx.\right]$