Math 3210 § 3. **Final Exam** Name:____________________

Treibergs Dec. 15, 2021

This is a closed book exam except that you are allowed two "cheat sheets," two 8.5"×11" pages with notes on both sides. Other notes, books, calculators, tablets, laptops, phones and text messaging devices are prohibited. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.	/21
2.	/22
3.	/21
4.	/22
5.	/22
6.	/21
7.	/22
8.	/22
9.	/20
Total	/150

1. (a) [10] Determine $c \geq 0$ so that the area under the graph of $f(x) = \dfrac{1}{1+x^3}$ from c to $2c$ is maximized.

   (b) [11] Determine whether the improper integral exists.

$$\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^4}$$

2. (a) [3] Let $f : (a, b) \to \mathbf{R}$. Define the *infimum of* f, $\inf\limits_{x\in(a,b)} f(x)$.

   (b) [19] Find $\inf\limits_{x\in(0,1)} \dfrac{1}{x+1}$ and prove your result.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] If $x, y > 0$ then $\log(xy) = \log x + \log y$. TRUE: ◯ FALSE: ◯

   (b) [7] If $f, g : [0,1] \to \mathbb{R}$ are bounded functions such that, fg is integrable on $[0,1]$, than at least one of f or g is integrable on $[0,1]$. TRUE: ◯ FALSE: ◯

   (c) [7] Let $f : \mathbf{R} \to \mathbf{R}$. Suppose both limits $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ exist. Then f is continuous at 0. TRUE: ◯ FALSE: ◯

4. (a) [3] Define what it means for $f : \mathbf{R} \to \mathbf{R}$ to be *differentiable* at the point $c \in \mathbf{R}$.

(b) [19] Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable at all $x \neq 0$ and that the limit

$$L = \lim_{x \to 0} f'(x)$$

exists and equals $L \in \mathbf{R}$. Show that f is differentiable at 0 and find $f'(0)$.

5. For $n \in \mathbf{N}$, let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions.

   (a) [3] Define: the sequence $\{a_n\}$ is a *Cauchy Sequence*.

   (b) [9] Prove using only the definition (a) that if $\{a_n\}$ is a Cauchy Sequence, then $\{a_n\}$ is bounded.

   (c) [10] Prove using only the definition (a) and the result (b) that if $\{a_n\}$ and $\{b_n\}$ are Cauchy Sequences, then $\{a_n b_n\}$ is a Cauchy Sequences.

6. (a) [3] Let $f, f_k : \mathbb{R} \to \mathbf{R}$. Define: $f(x) = \sum_{k=1}^{\infty} f_k(x)$ *converges uniformly* on $\mathbf{R}$.

(b) [18] Determine whether the series of functions $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly, where

$$f_k(x) = \begin{cases} \dfrac{x(k-x)}{k^4}, & \text{if } 0 \le x \le k; \\ 0, & \text{otherwise}. \end{cases}$$

7. Find the radius of convergence for each of the power series.

   (a) [7] $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{k+1}{3^k}(x-1)^k$

   (b) [7] $f(x) = \displaystyle\sum_{k=1}^{\infty} e^{k^2}(x-3)^k$

   (c) [8] $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}(x-4)^k$

8. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Complete the statement of the theorem. [Of several possible answers, select the one you prefer for part (b).]

   Theorem. *The bounded function f is integrable on $[a, b]$ if and only if*

   (b) [19] Using only the theorem in (b), show that $f(x)$ is integrable on $[-1, 1]$ where

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

9. For each infinite series, determine whether the series is absolutely convergent, convergent of divergent.

(a) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \log\left(\frac{k+1}{k}\right).$

ABSOLUTELY CONV.: ○ | CONDITIONALLY CONV.: ○ | DIVERGENT: ○

(b) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{2^k}{3^k+1}.$

ABSOLUTELY CONV.: ○ | CONDITIONALLY CONV.: ○ | DIVERGENT: ○

(c) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{\log k}{\log(k^2+k+1)}.$

ABSOLUTELY CONV.: ○ | CONDITIONALLY CONV.: ○ | DIVERGENT: ○

(d) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k^k}{(2k+1)!}.$

ABSOLUTELY CONV.: ○ | CONDITIONALLY CONV.: ○ | DIVERGENT: ○

(e) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k(k+2)}{(k+1)(k+3)(k+5)}.$

ABSOLUTELY CONV.: ○ | CONDITIONALLY CONV.: ○ | DIVERGENT: ○