Math 3070 § 1. Treibergs σπ

Second Midterm Exam

Name: Solutions

October 17, 2012

(1.) *In a 1997 study published in Ergonomics about offshore oil workers who took part in simulated escape exercise, it was reported that mean time to escape was 370.2 seconds and standard deviation of 24.00 seconds. Assume that these are the population mean and standard deviation of normally distributed escape times. What is the probability that the escape time is less than 360 sec? Give the answer to four decimal places. How long does it take for 67% of the workers to escape?*

Let X denote the escape time in seconds. We are told that X is normally distributed with $\mu = 370.2$ sec. and $\sigma = 24.00$ sec. Standardizing,

$$P(X < 360) = P\left(Z = \frac{X-\mu}{\sigma} < \frac{360.0 - 370.2}{24.00}\right) = \Phi(-.425).$$

Interpolating we find in Table A.3, $\Phi(-.43) = .3336$ and $\Phi(-.42) = .3372$. Since $-.425$ is midway between $(.5)(-.43) + (.5)(-.42) = -.425$, the interpolation is midway too $\Phi(-.425) \cong (.5)(.3336) + (.5)(.3372) = \boxed{.3354}$.

The 67%-ile is x_c such that $P(X < x_c) = .67$. We see in Table A.3 that $\Phi(.440) = .6700$. Thus the normalized percentile is $\eta(.67) = .440$. Hence 67% of the workers escape in

$$x_c = \mu + \eta(.67)\sigma = 370.2 + (.440)(24.0) = \boxed{380.76 \text{ sec.}}.$$

(2.) *Let $D = \{1, 2, 3, 4\}$. Let X be a random variable whose probability mass function is $p(x)$ for $x \in D$. Verify that $p(x)$ is a legitimate pmf. Compute $P(2 \le X \le 3)$. Compute $E(X)$. Let $h(x) = \dfrac{1}{x}$. Compute $E(h(X))$. Find the cumulative distribution function $F(x)$.*

$$p(x) = \begin{cases} \dfrac{x}{10}, & \text{if } x \in D; \\ 0, & \text{otherwise.} \end{cases}$$

To be a pdf, we need $p(x) \ge 0$ which holds since $p(x) = 0$ or $p(x) = x/10 > 0$ if $x = 1, 2, 3, 4$. We also need that the total probability be one. Here this holds because the total probability is

$$\sum_{x \in D} p(x) = p(1) + p(2) + p(3) + p(4) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1.$$

The probability

$$P(2 \le X \le 3) = P(X \in \{2, 3\}) = p(2) + p(3) = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \boxed{.5}.$$

The expectation is $E(X) = \sum_{x=1}^{4} x\, p(x)$ so

$$E(X) = 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10} = \frac{30}{10} = \boxed{3}.$$

The expectation of $h(X) = 1/X$ is $E(h(X)) = \sum_{x=1}^{4} h(x)\, p(x)$ so

$$E(h(X)) = \frac{1}{1} \cdot p(1) + \frac{1}{2} \cdot p(2) + \frac{1}{3} \cdot p(3) + \frac{1}{4} \cdot p(4) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \boxed{.4}.$$

The cumulative distribution function is $F(x) = \sum_{y \in D \text{ and } y \le x} p(y)$ so

$$\boxed{F(x) = \begin{cases} 0 & = 0 & = 0.0, & \text{if } x < 1; \\ p(1) & = .1 & = 0.1, & \text{if } 1 \le x < 2; \\ p(1)+p(2) & = .1+.2 & = 0.3, & \text{if } 2 \le x < 3; \\ p(1)+p(2)+p(3) & = .1+.2+.3 & = 0.6, & \text{if } 3 \le x < 4; \\ p(1)+p(2)+p(3)+p(4) & = .1+.2+.3+.4 & = 1.0, & \text{if } 4 \le x. \end{cases}}$$

(3.) *Mount Pleasant Company produces glassware. In their manufacturing process defects occur occasionally rendering pieces undesirable for marketing. It is known that, on average, 1 in every 2000 of these items produced have defects. What is the probability that a random sample of 8000 will yield fewer than 5 items possessing defects? What is the random variable here and how is it distributed? Write an exact expression for the answer. (You do not need to evaluate.) Find the probability using an approximation. Why is your approximation valid?*

Let X be the number of defects(successes) in a sample of 8000. This random variable is binomial with sample size $n = 8000$ and with a probability of defect(success) is $p = 1/2000 = .0005$. Then the exact probability of fewer than five defects is

$$P(X < 5) = P(X \le 4) = B(4, 8000, .005) = \boxed{\sum_{i=0}^{4} \binom{8000}{i} (.0005)^i (1 - .0005)^{8000-i}}.$$

In this case $\mu = np = 4$. We may approximate with the Poisson variables. From Table A.2,

$$P(X \le 4) \cong \text{Pois}(4, \mu = 4) = \boxed{.629}.$$

According to Devore's rule of thumb on P. 129, this is permissible if $8000 = n > 50$ and $4 = np < 5$, both of which are true.

(4.) *Let X be a random variable whose pdf X is given by $f(x)$. Identify this distribution as one of the standard distributions described in the text. What are the parameter values? What are μ and σ^2? Find the cumulative density function $F(x)$. Plot $F(x)$. Find the median $\tilde{\mu}$. For $0 < p < 1$, explain how you would find the pth quantile ((100p)th percentile), $\eta(p)$. From your graph, estimate $\eta(.2)$ and $\eta(.6)$.*

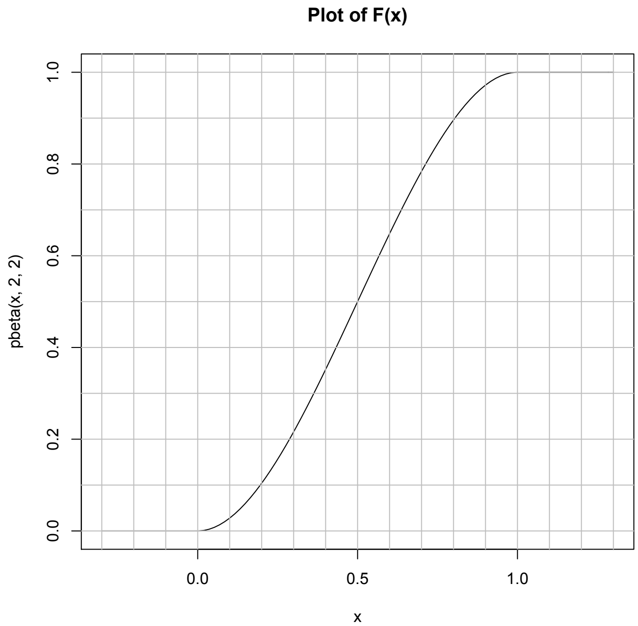
$$f(x) = \begin{cases} 6x(1-x), & \text{if } 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

This random variable X is distributed as the standard Beta Distribution (p. 176) with parameter values $A = 0$, $B = 1$, $\alpha = \beta = 2$. Note that the coefficient $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(2+2)}{\Gamma(2)\Gamma(2)} = \frac{3!}{1!\cdot 1!} = 6$. For the standard beta distribution, we know that

$$\mu = \frac{\alpha}{\alpha+\beta} = \frac{2}{2+2} = \boxed{\frac{1}{2}}, \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2\cdot 2}{(2+2)^2(2+2+1)} = \boxed{\frac{1}{20}}.$$

The cumulative distribution function is

$$F(x) = \boxed{\begin{cases} 0, & \text{if } x \le 0; \\ \int_0^x 6x(1-x)\,dx = 3x^2 - 2x^3, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } 1 \le x. \end{cases}}$$



Because $f(x)$ is symmetric about the $x = \frac{1}{2}$ line, the median is $\tilde{\mu} = \boxed{\frac{1}{2}}$. Otherwise, the median satisfies $\frac{1}{2} = F(\tilde{\mu}) = \tilde{\mu}^2(3 - 2\tilde{\mu})$. Because $F(\frac{1}{2}) = \frac{1}{2^2} \cdot (3 - 1) = \frac{1}{2}$ we have $\tilde{\mu} = \frac{1}{2}$.

From the graph, we find $\eta(p)$, the percentile from the equations $p = F(\eta(p))$. So that if $p = .2$ is the y-coordinate , then $\eta(.2) \cong \boxed{.3}$ is the corresponding x-coordinate. Similarly $\eta(.6) \cong \boxed{.56}$. The actual values are $\eta(.2) = 0.28714074$ and $\eta(.6) = 0.5670689$.

(5.) *Mount Pleasant Company is interested in evaluating its current inspection procedure on shipments of 50 goblets. The procedure is to take a sample of 5 from the shipment without replacement and pass the shipment if no more than 2 in the sample are found to be defective. What is the probability that the shipment will be passed if 10 of the goblets in the shipments are defective?*

Let X be the number of defective goblets in a sample of $n = 5$ selected without replacement from the shipment of $N = 50$. We assume that $M = 10$ is the number of defective goblets included in the 50. This is a hypergeometric variable $X \sim \text{Hypergeometric}(n = 5, M = 10, N = 50)$. Then using the hypergeometric pmf, the probability that the shipment is passed is

$$\begin{aligned} P(X \le 2) &= h(0, 5, 10, 50) + h(1, 5, 10, 50) + h(2, 5, 10, 50) \\ &= \frac{\binom{10}{0}\binom{40}{5}}{\binom{50}{5}} + \frac{\binom{10}{1}\binom{40}{4}}{\binom{50}{5}} + \frac{\binom{10}{2}\binom{40}{3}}{\binom{50}{5}} \\ &= \frac{1 \cdot 658008}{2118760} + \frac{10 \cdot 91390}{2118760} + \frac{45 \cdot 9880}{2118760} = \boxed{0.952}. \end{aligned}$$