Math 3070 § 1. **Cement Blocksd: Weibull PP Plot** Name: Example
Treibergs October 10, 2012

This demonstration illustrates using the PP-plot to check if data is reasonably distributed as Weibull. We use the data 6.9.8 provided in Navidi, *Statistics for Engineers and Scientists, 2nd. ed.,* Mc Graw Hill, 2008. The measurements are of the compressive strength in MPa of cement blocks that were cured for six days.

To generate a probability plot for Weibull variables, we follow the prescription in Devore, *Probability and Statistics for Engineers and Scientists, 5th ed.,* Duxbury, 2000, in section 4.6. Observe that the cumulative distribution function for Weibull random variable X with parameters $\alpha, \beta > 0$ is

$$\begin{aligned} P(X \le x) = F(x) &= 1 - \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) \\ &= 1 - \exp\left(-\exp\left(\alpha(\log x - \log\beta)\right)\right) \end{aligned}$$

After $u = \alpha(\log x - \log\beta)$, a linear change of variables, $\log x$ has the cumulative distribution

$$F(u) = 1 - e^{-e^{u}},$$

which is called the *extreme value distribution* because of other applications.

Thus we may plot quantiles of logarithm of the observed variable vs. the quantiles of the extreme value distribution to get a PP-plot for X. The observed quantiles are the sorted variables $\log(x_i)$, for $i = 1, 2, \ldots, n$. To get the theoretical quantiles, one solves $p_i = F(\eta(p_i))$ for the midpoints of equal intervals of $[0, 1]$ so for $i = 1, 2, \ldots, n$,

$$p_i = \frac{i - \frac{1}{2}}{n}$$

and so

$$q_i = \eta(p_i) = \log\bigl(-\log(1 - p_i)\bigr).$$

The Weibull PP-plot consists of the points $(q_i, \log(x_i))$.

After we're convinced that Weibull is a reasonable distribution for our data, how do we determine the parameters α and β? We use the method of moments, which is just a fancy way of saying we pick the Weibull parameters in such a way that their mean and variance is the sample mean and sample variance of the data. For Weibull variables, the population mean and variance are given by

$$\mu = \beta\Gamma\left(1 + \frac{1}{\alpha}\right),$$

$$\sigma^2 = \beta^2\left\{\Gamma\left(2 + \frac{1}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right\}$$

By adding the square of the mean, we isolate α.

$$\frac{\sigma^2 + \mu^2}{\mu^2} = \frac{\Gamma\left(1 + \frac{2}{\alpha}\right)}{\left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2} = g(\alpha)$$

Using the sample statistics, we solve for α in

$$r = \frac{s^2 + \bar{x}^2}{\bar{x}^2} = g(\alpha).$$

Once α is known, we solve the equation for the mean for β:

$$\beta = \frac{\bar{x}}{\Gamma\left(1 + \frac{1}{\alpha}\right)}.$$

One way to solve $r = g(\alpha)$ for α is to use the root finder `uniroot()` in **R**©. By plotting, we see that $g(\alpha)$ is a strictly decreasing function. We observe that $g(1) > r > g(100)$ so that we supply the interval for the root finder

```
uniroot(g(x)-r,c(1,100))
```

which spits out the root α and its accuracy.

R Session:

```
R version 2.10.1 (2009-12-14)
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[R.app GUI 1.31 (5538) powerpc-apple-darwin8.11.1]



> ########## READ THE COMPRESSIVE STRENGTH DATA (NAVIDI 6.9.8) ###############
> tii <- scan()
1: 1387 1301 1376 1397 1399 1378 1343 1349 1321 1364 1332 1396 1372
14: 1341 1374
16:
Read 15 items
> tii
 [1] 1387 1301 1376 1397 1399 1378 1343 1349 1321 1364 1332 1396 1372
[14] 1341 1374

> n <- length(tii);n
[1] 15

> # Sort
> sii <- sort(tii); sii
 [1] 1301 1321 1332 1341 1343 1349 1364 1372 1374 1376 1378 1387 1396
[14] 1397 1399
```
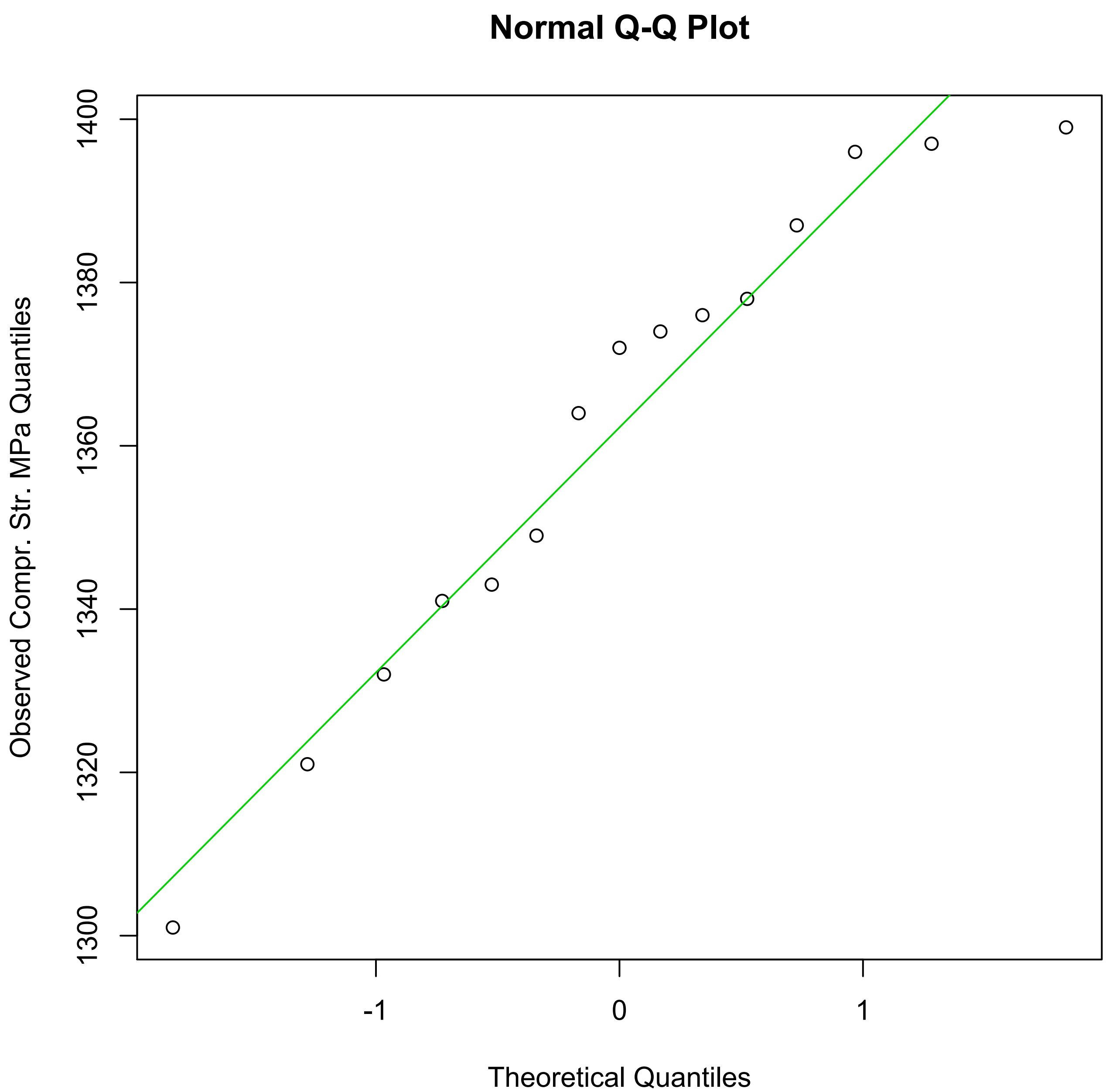
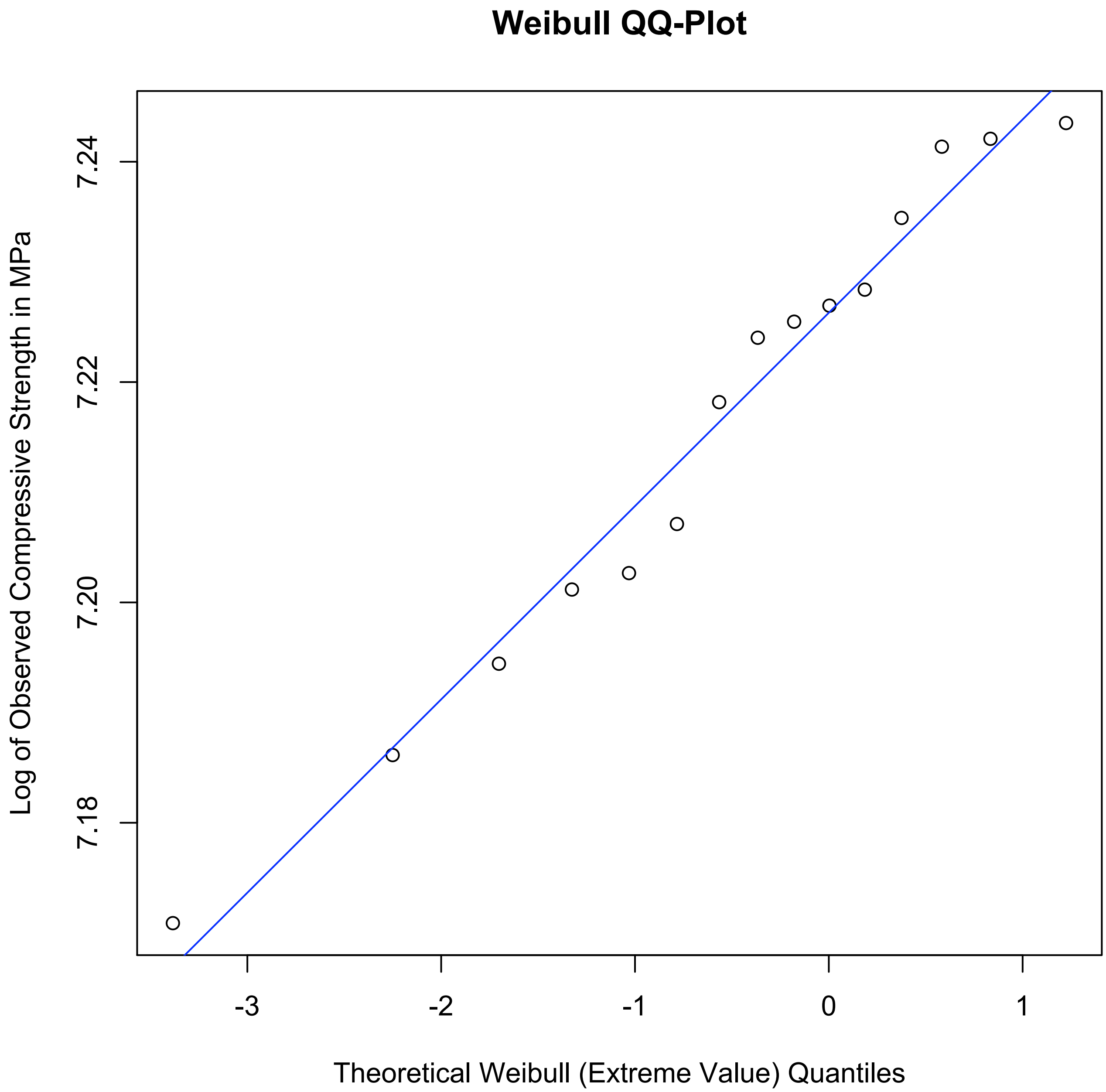

```
> # Take logs
> lsii <- log(sii); lsii
 [1] 7.170888 7.186144 7.194437 7.201171 7.202661 7.207119 7.218177
 [8] 7.224025 7.225481 7.226936 7.228388 7.234898 7.241366 7.242082
[15] 7.243513

> ################# CHECK IF DATA IS REASONABLY NORMALLY DISTRIBUTED ##########
> qqnorm(tii,ylab="Observed Compr. Str. MPa Quantiles")
> qqline(tii,col=3)
> # M3075Cement2.pdf
> # The plot is somewhat "r" shaped, so the normality may not hold. Although
> # n=15 is small and the normal QQ-Plot does not strongly indicate non-normal.
> # Here we suppose that there is strong engineering reason to suspect Weibull.
```

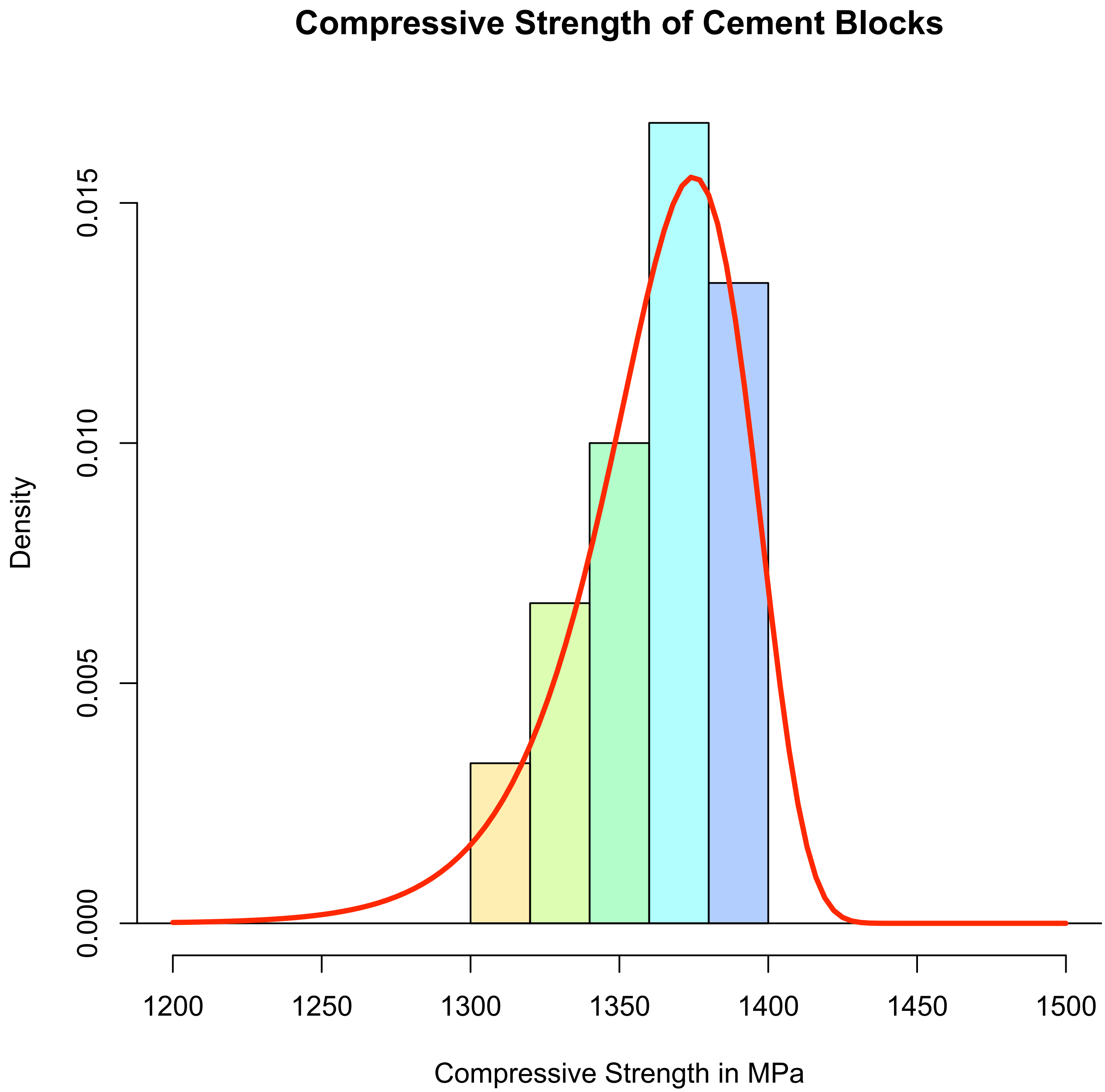

Normal Q-Q Plot
Observed Compr. Str. MPa Quantiles
1300
1320
1340
1360
1380
1400
-1
0
1
Theoretical Quantiles

```
> ##################### CONSTRUCT THE THEORETICAL QUANTILES ##################
> # "ppoints" gives midpoints of  n  equal intervals of [0,1]
> q <- ppoints(n);q
 [1] 0.03333333 0.10000000 0.16666667 0.23333333 0.30000000 0.36666667
 [7] 0.43333333 0.50000000 0.56666667 0.63333333 0.70000000 0.76666667
[13] 0.83333333 0.90000000 0.96666667

> # For Weibull, the quantiles (Extreme value Quantiles)
> qq <- log(-log(1-q)); qq
 [1] -3.384294493 -2.250367327 -1.701983355 -1.325375512 -1.030930433
 [6] -0.783600688 -0.565661963 -0.366512921 -0.178830030  0.003296669
[11]  0.185626759  0.375203292  0.583198081  0.834032445  1.224127541

> qq <- log(-log(1-pp))
>
> ############# TEST IF DATA IS WEIBULL USING WEIBULL QQ-PLOT ################
>
> qqplot(qq,lsii,main="Weibull QQ-Plot",ylab="Log of Observed Compressive
  Strength in MPa",xlab="Theoretical Weibull (Extreme Value) Quantiles")
> abline(lm(lsii ~ qq),col=4)
> # M3075Cemen3.pdf
> # The points line up better. Thus Weibull is a reasonable distribution.
```


Weibull QQ-Plot
Log of Observed Compressive Strength in MPa
7.18
7.20
7.22
7.24
-3
-2
-1
0
1
Theoretical Weibull (Extreme Value) Quantiles

```
>
> ############### FIND THE PARAMETERS OF THE WEIBULL DISTRIBUTION ############
>
> # We shall use the method of moments. We compute the sample mean  m  and
> # sample variance  v  and find  Weibull  alpha and beta that have
> # mu = m  and  sigma^2 = v.
>
> m<- mean(tii)
> v <- var(tii)
> ratio <- (v+m^2)/m^2;ratio
[1] 1.000475
> # define a function of alpha
> f <- function(x){gamma(1+2/x)/gamma(1+1/x)^2}
> #  f  is a decreasing function: see it by plotting
> curve(f(x),1,10)
> #
> # To solve  f(alpha)=ratio, we need to find an interval that brackets
> f(1)
[1] 2
> f(10)
[1] 1.014475
> f(100)
[1] 1.000162
> # Thus F(1) > ratio > f(100).
> # "uniroot" solves a one variable equation f(x)-ratio=0
>
> uniroot(function(x){gamma(1+2/x)/gamma(1+1/x)^2-ratio},c(1,100))
$root
[1] 58.10109

$f.root
[1] -1.074045e-10

$iter
[1] 8

$estim.prec
[1] 6.103516e-05

> alpha<- 58.10109
> beta <- m/gamma(1+1/alpha);beta
[1] 1375.266
>
> # Check that these alpha and beta give the mean and variance
> beta*gamma(1+1/alpha)
[1] 1362
> # same as  m
>
> beta^2*(gamma(1+2/alpha)-gamma(1+1/alpha)^2)
[1] 881.9998
> # almost the same as  v
```

```
> ################ PLOT THE DENSITY FUNCTION ON TOP OF THE HISTOGRAM #######
>
> hist(tii,freq=F,xlim=c(1200,1500),col=rainbow(8,alpha=.3)[2:6], main=
  "Compressive Strength of Cement Blocks", xlab="Compressive Strength in MPa")
> abline(h=0)
> curve(dweibull(x,scale=beta,shape=alpha),add=T,col=2,lwd=3)
> # M3075Cemen1.pdf
>
```



Compressive Strength of Cement Blocks
Density
0.000
0.005
0.010
0.015
1200
1250
1300
1350
1400
1450
1500
Compressive Strength in MPa