Math 3070 § 1. Treibergs | **Maximum Estimator Example: Simulation of Confidence Intervals** | Name: Example July 9, 2011

The maximum estimator is introduced in Problem 7.59 of Devore, *Probability and Statistics for Engineering and the Sciences, 8th ed.,* Brooks Cole, 2012. Devore applies it to fake data that is supposed to be waiting times for a bus.

Assume that the pdf is uniform depending on the parameter $\theta > 0$,

$$f(x) = \begin{cases} \dfrac{1}{\theta}, & \text{if } 0 \le x \le \theta; \\ 0, & \text{otherwise.} \end{cases}$$

Integrating gives the cdf

$$F(x) = \begin{cases} 0, & \text{if } x \le 0; \\ \dfrac{x}{\theta}, & \text{if } 0 < x \le \theta; \\ 1, & \text{if } \theta < x. \end{cases}$$

An estimator for θ is the maximum of the sample $X_1, X_2, \ldots, X_n \sim \mathrm{Unif}(0, \theta)$, but as we have seen, it is biased. An unbiased estimator is given by

$$\hat{\theta} = \frac{n+1}{n} \max(X_1, X_2, \ldots, X_n).$$

To see this, let us recall the derivation of its pdf. The cdf, by independence

$$\begin{aligned} F(x) &= \mathrm{P}\left(\hat{\theta} \le x\right) \\ &= \mathrm{P}\left(\frac{n+1}{n} \max(X_1, X_2, \ldots, X_n) \le x\right) \\ &= \mathrm{P}\left(\left\{X_1 \le \frac{nx}{n+1}\right\} \cap \left\{X_2 \le \frac{nx}{n+1}\right\} \cap \cdots \cap \left\{X_n \le \frac{nx}{n+1}\right\}\right) \\ &= \mathrm{P}\left(X_1 \le \frac{nx}{n+1}\right) \cdot \mathrm{P}\left(X_2 \le \frac{nx}{n+1}\right) \cdots \mathrm{P}\left(X_n \le \frac{nx}{n+1}\right) \\ &= F\left(\frac{nx}{n+1}\right)^n \\ &= \begin{cases} 0, & \text{if } x \le 0; \\ \left(\dfrac{nx}{(n+1)\theta}\right)^n, & \text{if } 0 < x \le \dfrac{n+1}{n}\theta; \\ 1, & \text{if } \dfrac{n+1}{n}\theta < x. \end{cases} \end{aligned}$$

The pdf for $\hat{\theta}$ is thus

$$f_{\hat{\theta}}(x) = \frac{d}{dx} F(x) = \begin{cases} \dfrac{n^{n+1} x^{n-1}}{(n+1)^n \theta^n}, & \text{if } 0 \le x \le \dfrac{n+1}{n}\theta; \\ 0, & \text{otherwise.} \end{cases}$$

Let $0 < \alpha < 1$. Let us compute the probability

$$\begin{aligned}
\mathrm{P}\left(\theta\left(\frac{\alpha}{2}\right)^{1/n} \le \max(X_1, X_2, \ldots, X_n) \le \theta\left(1-\frac{\alpha}{2}\right)^{1/n}\right) \\
&= \mathrm{P}\left(\theta\left(\frac{\alpha}{2}\right)^{1/n} \le \frac{n\hat\theta}{n+1} \le \theta\left(1-\frac{\alpha}{2}\right)^{1/n}\right) \\
&= F\left(\theta\left(\frac{\alpha}{2}\right)^{1/n}\right) - F\left(\theta\left(1-\frac{\alpha}{2}\right)^{1/n}\right) \\
&= \left(\frac{\alpha}{2}\right) - \left(1-\frac{\alpha}{2}\right) \\
&= 1-\alpha.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{P}\left(\theta\alpha^{1/n} \le \frac{n\hat\theta}{n+1} = \max(X_1, X_2, \ldots, X_n) \le \theta\right) &= F(\theta) - F\left(\theta\alpha^{1/n}\right) \\
&= 1-\alpha.
\end{aligned}$$

These two probability statements become confidence intervals by solving the inequalities for θ. With $1-\alpha$ confidence, we obtain

$$\frac{\max(X_1, X_2, \ldots, X_n)}{\left(1-\frac{\alpha}{2}\right)^{1/n}} \le \theta \le \left(\frac{2}{\alpha}\right)^{1/n} \max(X_1, X_2, \ldots, X_n) \tag{1}$$

$$\max(X_1, X_2, \ldots, X_n) \le \theta \le \frac{\max(X_1, X_2, \ldots, X_n)}{\alpha^{1/n}} \tag{2}$$

The gap in inequality (1) exceeds that in (2) because $2^{1/n} > 1$ and $2-\alpha > 1$ imply

$$\frac{1}{\alpha^{1/n}} - 1 < 2^{1/n}\left[\frac{1}{\alpha^{1/n}} - \frac{1}{(2-\alpha)^{1/n}}\right] = \left(\frac{2}{\alpha}\right)^{1/n} - \frac{1}{\left(1-\frac{\alpha}{2}\right)^{1/n}}$$

We plot both widths, the left vs. right sides of the inequality.

We check that the estimator $\hat\theta$ is unbiased. Note that for sample from $\mathrm{Unif}(0,\theta)$, the maximum of all X_i is θ so that the statistic ranges in $0 \le \hat\theta \le \frac{n+1}{n}\theta$. Thus

$$\begin{aligned}
\mathrm{E}(\hat\theta) &= \int_0^{\frac{n+1}{n}\theta} u\, f_{\hat\theta}(u)\, du \\
&= \int_0^{\frac{n+1}{n}\theta} \frac{n^{n+1}u^n}{(n+1)^n\theta^n}\, du \\
&= \left[\frac{n^{n+1}u^{n+1}}{(n+1)^{n+1}\theta^n}\right]_0^{\frac{n+1}{n}\theta} \\
&= \theta.
\end{aligned}$$

We simulate $B = 10,000$ samples of size $n = 5$ taken from $\mathrm{Unif}(0,1)$. We plot a histogram of the resulting $\hat\theta$'s and superimpose the densith $f_{\hat\theta}$.

We also simulate $B = 100$ samples of size $n = 5$ taken from $\mathrm{Unif}(0,1)$ and plot the corresponding $\alpha = .05$ confidence intervals using (2). It turned out that six of these did not capture the $\theta = 1$, which is about the expected 5%.

R Session:

```
R version 2.10.1 (2009-12-14)
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'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.31 (5538) powerpc-apple-darwin8.11.1]

[Workspace restored from /Users/andrejstreibergs/.RData]

> ################# COMPUTE THE TWO CI'S FOR THETA ############################
>
> # two ci for th of dist unif(0,th)
> n <- 5
> alpha <- .05
> l1 <- function(y){max(y)*(1-alpha/2)^(-1/n)}
> u1 <- function(y){max(y)*(alpha/2)^(-1/n)}
> l2 <- function(y){max(y)}
> u2 <- function(y){max(y)*(alpha)^(-1/n)}
>
> ################# ENDTER DEVORE'S FAKE TIMES FROM EX. 7.59 ##################
> x <- scan()
1: 4.2  3.5 1.7 1.2 2.4
6:
Read 5 items
> # First CI and its width.
> c(l1(x),u1(x));u1(x)-l1(x)
[1] 4.221321 8.783372
[1] 4.562051
> # Second CI and its width.
> c(l2(x),u2(x));u2(x)-l2(x)
[1] 4.20000 7.64637
[1] 3.446370
```
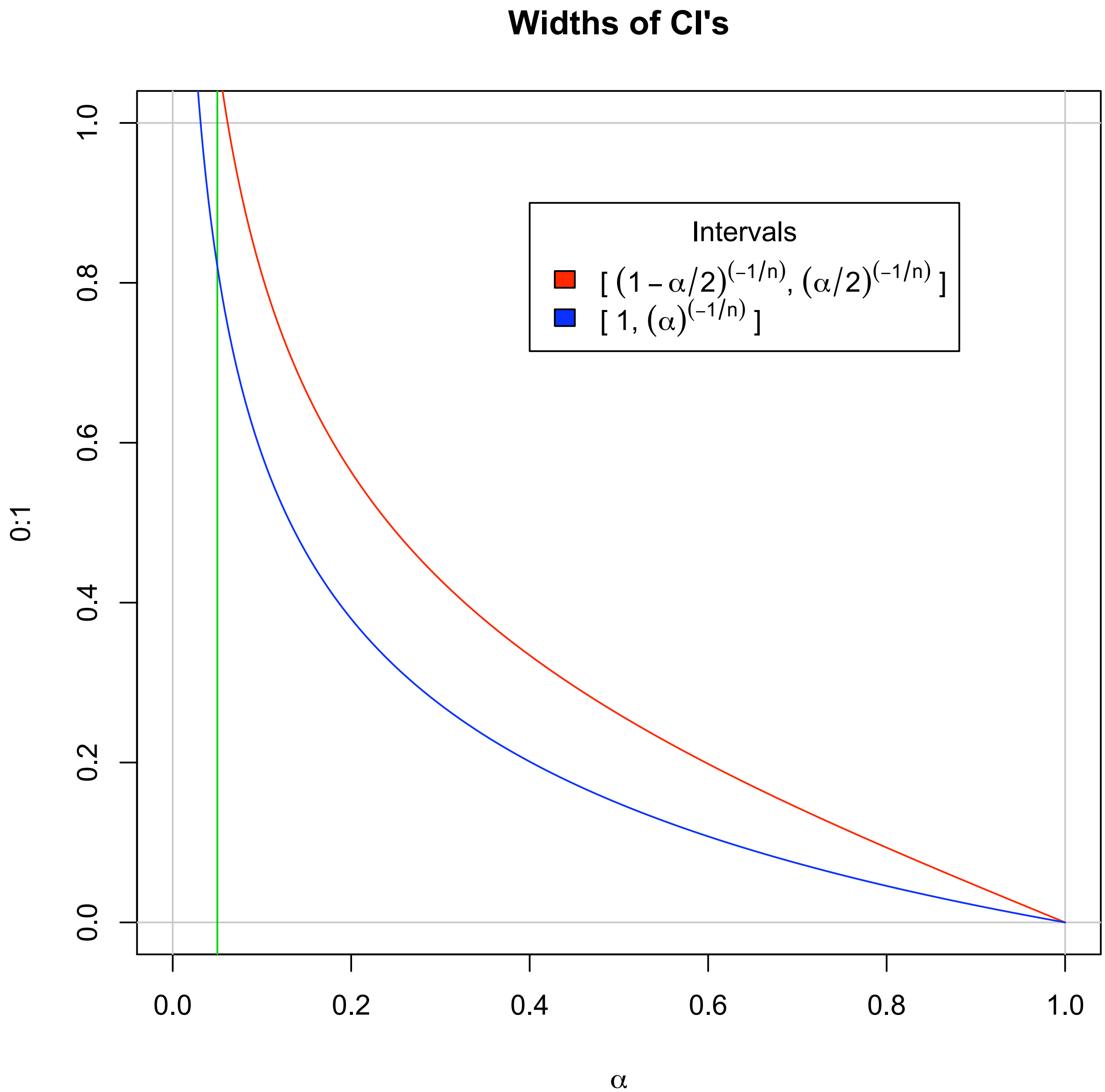
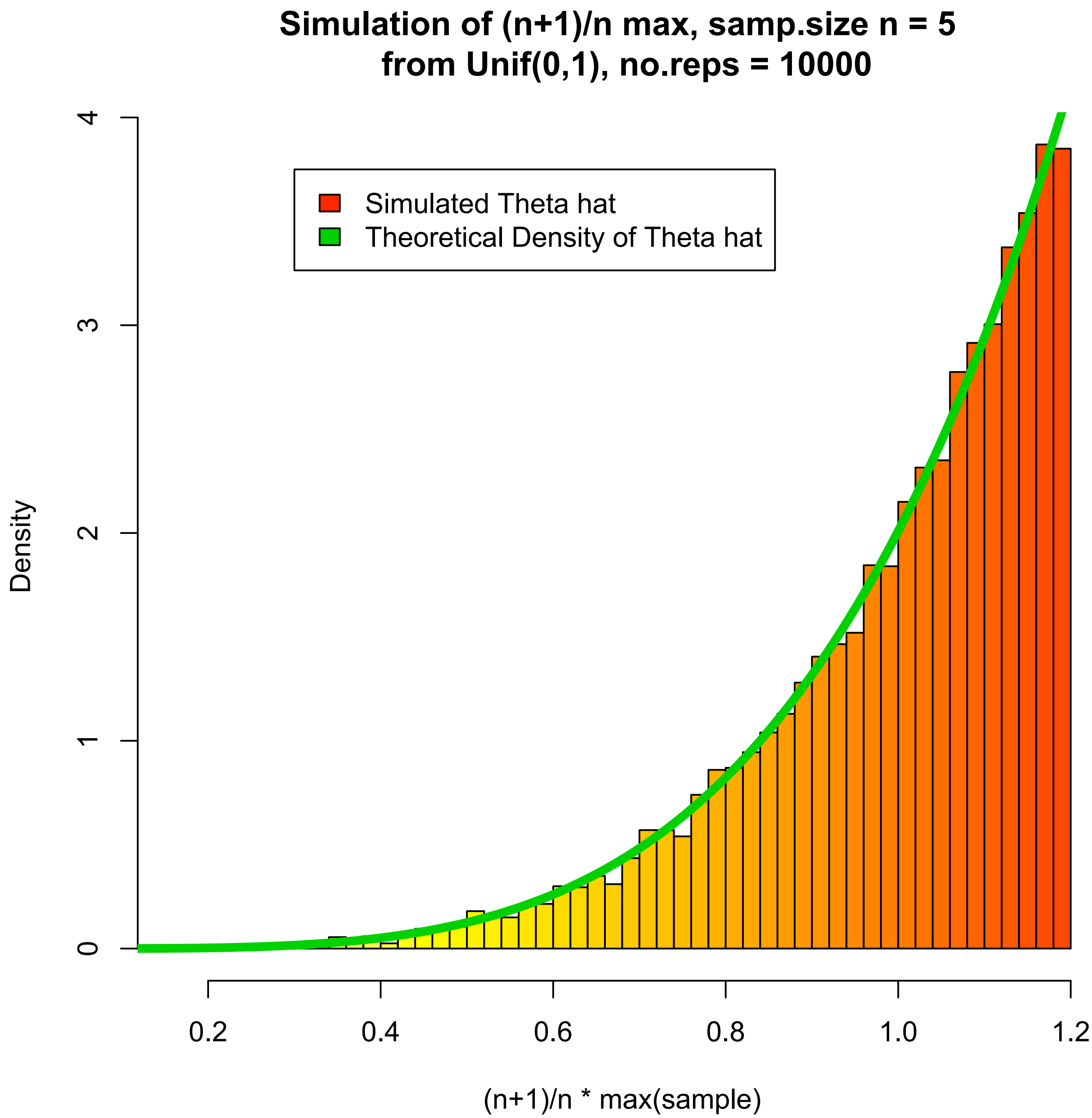
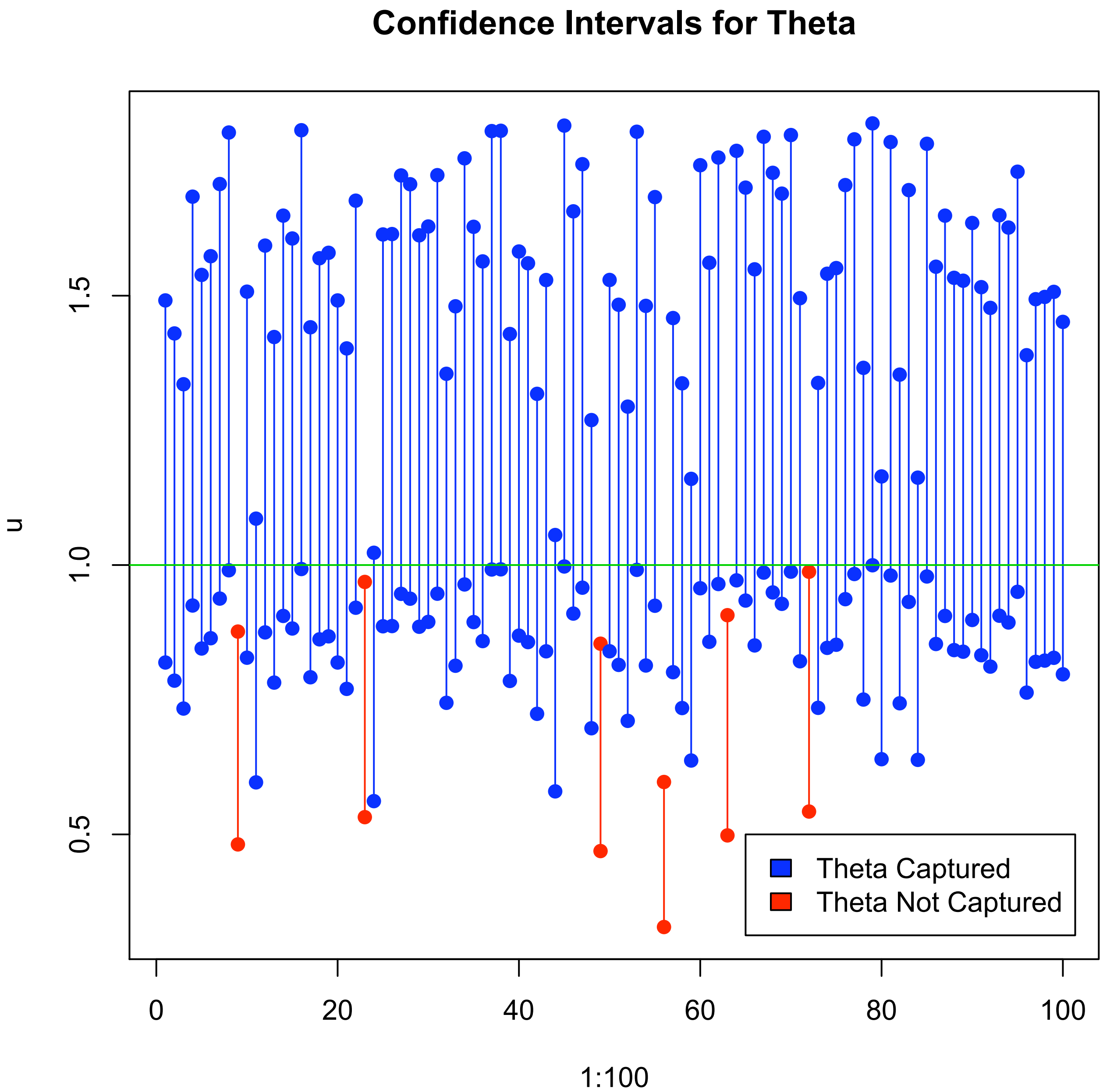

```
> ################## PLOT TWO WIDTHS TO COMPARE #############################
> # Interval width as fn of alpha
> w1 <- function(a){(a/2)^(-1/n)-(1-a/2)^(-1/n)}
> w2 <- function(a){(a)^(-1/n)-1}
> xs <- seq(0,1,1/377)
> plot(0:1, xlim = 0:1, type = "n", ylim = 0:1, main = "Widths of CI's",
+ xlab = expression(alpha))
> abline(h = 0:1, col=8); abline(v = 0:1, col=8); abline(v = alpha, col = 3)
> lines(xs,w1(xs),col=2)
> lines(xs,w2(xs),col=4)
> legend(.4, .9, legend = c(expression(paste("[ ", (1-alpha/2)^(-1/n), ", ",
+ (alpha/2)^(-1/n), " ]")), expression(paste("[ 1, ", (alpha)^(-1/n) ," ]" ))),
+ fill = c(2,4), bg="white", title = "Intervals")
> # M3074MaxUnifCI1.pdf

> ################## SIMULATE SAMPLING DISTRIBUTION OF THETA HAT ############
>
> B <- 10000
> n <- 5
> c <- (n+1)/n
> v <- replicate(B, c*max(runif(n, 0, 1)))
> hist(v,breaks=40, col=rev(heat.colors(50)))
> hist(v,breaks=40, col=rev(heat.colors(60)))
> summary(v); sd(v)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 0.1720  0.9111  1.0460  1.0010  1.1320  1.2000
[1] 0.1685048

> # Plot the histogram.
> hist(v, breaks = 40, col = rev(heat.colors(60)), xlab="(n+1)/n * max(sample)",
+ main = paste("Simulation of (n+1)/n max, samp.size n =", n,
+ "\n from Unif(0,1), no.reps =", B), freq = F)
> lines(xs, n/c*(xs/c)^(n-1), col = 3, lwd = 5)
> legend(.3, 3.75, legend = c("Simulated Theta hat",
+ "Theoretical Density of Theta hat"), fill = c(2, 3))
> # M3074MaxUnifCI2.pdf
```


Widths of CI's
Intervals
[$(1-\alpha/2)^{(-1/n)}$, $(\alpha/2)^{(-1/n)}$]
[1, $(\alpha)^{(-1/n)}$]
0:1
α
0.0
0.2
0.4
0.6
0.8
1.0


Simulation of (n+1)/n max, samp.size n = 5
from Unif(0,1), no.reps = 10000
Simulated Theta hat
Theoretical Density of Theta hat
Density
0
1
2
3
4
0.2
0.4
0.6
0.8
1.0
1.2
(n+1)/n * max(sample)

```
> ################## SIMULATE 100 CI'S FOR THETA ##############################
>
> u <- numeric(100);l <- numeric(100)
> # Fill color vector with blues.
> cl <- rep(4,100)
> # Generate CI's. Test if  theta  is captured.
> for(i in 1:100)
+               {
+                sam <- runif(n,0,1)
+                l[i] <- l2(sam)
+                u[i] <- u2(sam);
+                if(l[i] > 1 | u[i] < 1)
+                                      {
+                                       cl[i]<-2
+                                      }
+               }
> mx <- max(u)
> mi <- min(l)
> plot(1:100, u, ylim = c(mi, mx), type = "n",
+ main = "Confidence Intervals for Theta")
> for(i in 1:100)
+               {
+                points(c(i, i), c(l[i], u[i]), col = cl[i], pch = 19)
+                lines(c(i, i), c(l[i], u[i]), col = cl[i])
+               }
> abline(h = 1, col = 3)
> legend(65, .5, legend = c("Theta Captured", "Theta Not Captured"),
+ fill = c(4, 2), bg = "white")
> # M3074MaxUnifCI3.pdf
```


Confidence Intervals for Theta
u
1.5
1.0
0.5
0
20
40
60
80
100
1:100
Theta Captured
Theta Not Captured