Math 3070 § 1.   **Second Midterm Exam**   Name:_____Solutions_____
Treibergs $\sigma\tau$ February 27, 2008

1. *The Parowan Parts company received a lot that contains 500 items, 100 of which are defective. Suppose that 20 items are selected at random without replacement. Write an exact expression for the probability that fewer than three items are defective. (You do not need to evaluate your formula.) Use an approximation to compute the probability. Why is your approximation valid?*

Since selection is without replacement, the number of defective parts X is a hypergeometric random variable taken from a sample of size $n = 20$ drawn from a population of size $N = 500$ of which $M = 100$ are successes (flawed). The exact probability that two or fewer are flawed is
$P(X \leq 2) = h(0; 20, 100, 500) + h(1; 20, 100, 500) + h(2; 20, 100, 500) =$

$$P(X \leq 2) = \frac{\binom{100}{0}\binom{400}{20}}{\binom{500}{20}} + \frac{\binom{100}{1}\binom{400}{19}}{\binom{500}{20}} + \frac{\binom{100}{2}\binom{400}{18}}{\binom{500}{20}}$$

According to the rule of thumb, since $n/N = 20/500 = .04 \leq .05$ it is possible to approximate by binomial variables with probability of success $p = M/N = 100/500 = .2$. From Table A1 on p. 665, $P(X \leq 2) \approx B(2; 20, .2) = \boxed{.206}$.

2. *The lifetimes of economy lightbulbs available at the home center store are known to be normally distributed with a mean of 737. hours and a standard deviation of 38.2 hours. Answer the following questions to three decimal places. What is the probability that a random light bulb lasts more than 750. hours? Find the number of hours T so that 70.0% of all lightbulbs last longer than T.*

Standardizing to $Z = (X - \mu)/\sigma$, using table A3 on p. 669, $P(X > 750.) = 1 - P(X \leq 750.) = 1 - P\left(Z \leq \frac{750. - 737.}{38.2}\right) = 1 - \Phi(.3403) = 1 - .6331 = \boxed{.367}$.

If T is the number of hours such that 30.0% of lifetimes are shorter, by putting $z_c = \frac{T - 737.}{38.2}$ we have $.300 = P(X \leq T) = \Phi(z_c)$. Interpolating in Table A3, we have $\Phi(-.530) = .2981$ and $\Phi(-.520) = .3015$ so $(.3000 - .2981)/(.3015 - .2981) = .559$ so
$z_c = -.530 + (.559)[(-.520) - (-.530)] = -.524$. Thus $T = 38.2 z_c + 737. = \boxed{717.}$

3. *Suppose that the number of flaws in a fiber optic cable is a Poisson process with an average of* 0.6 *per* 100 *feet. What is the probability of at most two flaws in a* 200 *foot cable?*

The Poisson random variable has a mean of $\lambda = \alpha T = 0.6$ in a length of $T = 100$ feet so $\alpha = .006$. Thus in 200 feet, $\lambda = (.006)(200) = 1.2$. Since the $\lambda = 1.2$ cdf is not tabulated, the probability of two or fewer flaws is $F(2, \lambda) =$

$$= p(0; \lambda) + p(1; \lambda) + p(2; \lambda) = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} = e^{-1.2}\left(1 + 1.2 + \frac{1.2^2}{2}\right) = \boxed{.879}$$

4. *In Davis County, the proportion of highway sections needing repairs in any given year is a random variable having the beta distribution with $\alpha = 3$ and $\beta = 2$. On average, what percentage of highway sections will require repair in any given year? Find the probability that at most half of the highway sections will require repair on any given year.*

Let X be the proportion of highway sections needing repair in a random year (so $A = 0$ and $B = 1$.) The formula for the mean of a standard beta variable says on average, $\mu = \frac{\alpha}{\alpha+\beta} = \frac{3}{3+2} = .600$ of the highway sections need repair any given year. The density function is
$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{4!}{2!1!} x^2(1-x) = 12(x^2 - x^3).$$
Thus, the probability that at most half the streets will need repair is
$$P(X \leq 0.5) = \int_0^{1/2} 12(x^2 - x^3)\, dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{1/2} = 12\left[\frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4}\right] = \frac{1}{2} - \frac{3}{16} = \boxed{\frac{5}{16}}.$$

5.) *Find the constant c that makes $f(x)$ a probability density function. $f(x) = cx^{-4}$ if $x \geq 2$ and* 0 *otherwise. What is the variance of X?*

The total probability is $1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_2^{\infty} cx^{-4}\, dx = c\left[-\frac{x^{-3}}{3}\right]_2^{\infty} = \frac{c}{24}$ so $c = 24$.

The expected value is $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_2^{\infty} 24x^{-3}\, dx = 24\left[-\frac{x^{-2}}{2}\right]_2^{\infty} = \frac{24}{8} = 3$. The expectation of the square is $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_2^{\infty} 24x^{-2}\, dx = 24\left[-x^{-1}\right]_2^{\infty} = \frac{24}{2} = 12$.

Thus the variance is $V(X) = E(X^2) - [E(X)]^2 = 12 - 3^2 = \boxed{3}$.