Math 3010 § 1. **First Midterm Exam** Name: Solutions
Treibergs February 4, 2026

1. *For each mathematician state as precisely as you can a theorem or teaching attributed to this mathematician.*

Mathematician	Theorem or Teaching
Thales 622 – 547 BC Miletus	First Greek to have written proofs and have a theorem named after him. Th. In an isosceles triangle opposite angles are equal.
Pythagoras 585 – 500 BC Croton	Stressed geometry, arithmetic, astronomy and music. "All is number." Figurate numbers. Th. $\sqrt{2}$ is irrational.
Hippocrates 460 – 380 BC Chios	Wrote early logical treatise about geometry. Th. The quadrature of the lune.
Eudoxus 408 – 355 BC Cnidus	Resolved Xeno's paradox. Wrote on proportion, exhaustion. Th. Area of a circle is proportional to diameter squared.
Aristotle 384 – 322 BC Athens	Taught a theory of rigorous argument via axiomatic approach. Described syllogisms.
Euclid 323 – 285 BC Alexandria	His 13 book "Elements" summarized mathematics. An influential model of logic and deductive method. He gave several proofs of Pythagorean Theorem.
Theon 65 – 135 AD Smyrna	His edited versions of Euclid survive. Wrote on number theory and geometry. Th. Sequence of side and diagonal numbers approximate $\sqrt{2}$.

2. (a) *Use the Babylonian method and compute using sexagesimal arithmetic to find the quotient. (Other methods receive zero credit.)*

$33, 44 \quad \div \quad 5 \quad = \quad ?$

Babylonians multiplied by the reciprocal expressed in sexagesimal. Thus

$$\frac{1}{5} = \frac{12}{60} =; 12.$$

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 33,  44                            33, 44  =  33 x 60  +  44  =  2024
x         ;   12
----------------
     396  ;  528
----------------
  6,   8  ;                            528  =   8 x 60  +  48
      36  ;   48                       396  =   6 x 60  +  36
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  6,  44  ;   48
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Check: $\frac{2024}{5} = 404\frac{4}{5} = 6 \times 60 + 44 + \frac{48}{60} = 6, 44; 48.$

(b) *Using the Egyptian method of doubling, find the quotient. Make sure that your solution is expressed with a proper Egyptian unit fraction. (Other methods receive zero credit.)*

$222 \quad \div \quad 11 \quad = \quad ?$

1	$\times$	11	=	11	
2	$\times$	11	=	22	
4	$\times$	11	=	44	$\surd$
8	$\times$	11	=	88	
16	$\times$	11	=	176	$\surd$
$\overline{2}$	$\times$	11	=	$5\ \overline{2}$	
$\overline{4}$	$\times$	11	=	$2\ \overline{2}\ \overline{4}$	
$\overline{8}$	$\times$	11	=	$1\ \overline{4}\ \overline{8}$	$\surd$
$\overline{11}$	$\times$	33	=	1	
$\overline{22}$	$\times$	33	=	$\overline{2}$	$\surd$
$\overline{44}$	$\times$	33	=	$\overline{4}$	
$\overline{88}$	$\times$	33	=	$\overline{8}$	$\surd$

Subtracting from the the right column until zero is left and putting a check mark at the corresponding row

$$
\begin{array}{r}
222 \\
\underline{-176} \\
46 \\
\underline{-44} \\
2 \\
\underline{-\quad 1\ \overline{4}\ \overline{8}} \\
\overline{2}\ \overline{8} \\
\underline{-\overline{2}} \\
\overline{8} \\
\underline{-\overline{8}} \\
0
\end{array}
$$

Thus the quotient is the sum from the checked rows

$$q = 16 + 4 + \overline{8} + \overline{22} + \overline{88} = 20\ \overline{8}\ \overline{22}\ \overline{88}.$$

An alternative expression is gotten by using

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

so with $n = 11$,

$$\frac{2}{11} = \frac{1}{11} + \frac{1}{11} = \frac{1}{11} + \frac{1}{12} + \frac{1}{132},$$

which yields

$$\frac{222}{11} = 20\frac{2}{11} = 20\ \overline{11}\ \overline{12}\ \overline{132}.$$

Curiously, the Egyptian tables use

$$\frac{2}{11} = \frac{12}{66} + \frac{11+1}{66} = \frac{1}{6} + \frac{1}{66} = \ \overline{6}\ \overline{66}.$$

3. (a) *Determine whether the following statements are true or false.*

   i. Statement: *The Babylonians knew infinitely many essentially different Pythagorean Triples.*
   True.

   ii. Statement: *The Egyptians had a formula for the area of a circle whose diameter is $D = 45$.*
   True.

   iii. Statement: *The Pythagoreans had difficulty believing that Achilles catches up to the tortoise in Zeno's Paradox.*
   True.

   iv. Statement: *The Greeks could bisect an angle using just straightedge and compass.*
   True.

   (b) *Give a detailed explanantion of ONE of your answers (i)–(iv) above.*

i. The large cuneiform tablet Plimpton 322 lists 15 huge Pythagorean triples as discussed in section 2.6. Burton argues that the Babylonians must have had general formulas to generate the entries in the table. With choices of increasing m and n, the entries follow the formulae

$$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2$$

for which $x^2 + y^2 = z^2$. This scheme gives infinitely many Pythagorean triples.

ii. The Egyptians used the formula

$$A = \left(D - \frac{1}{9}D\right)^2$$

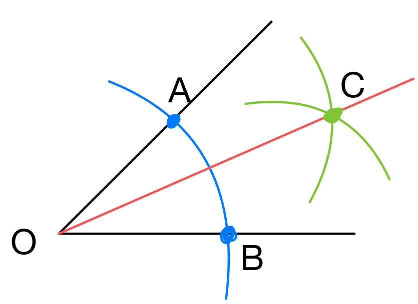
where D is the diameter of the circle. In this case, $d = 45$ so

$$A = \left(45 - \frac{1}{9} \cdot 45\right)^2 = (45 - 5)^2 = 40^2 = 1600.$$

iii. Achilles and the Tortoise. Achilles and a tortoise decided to have a 100 meter race. Because Achilles runs faster he offered to give the tortoise a head start. The tortoise said that with a one meter head start, Achilles would never overtake him. Because in the time that Achilles catches up to where the tortoise started, the tortoise would have moved on. And then when Achilles traverses the distance to the tortoise is now, it would have moved ahead. And so on. Whenever Achilles gets to where the tortoise has been, it will have moved on, and Achilles would never be able to overtake the tortoise.

"Zeno pointed out the absurdity of the concept of 'infinite dividibility' of time and space. ... Zeno's argument confounded his contemporaries. ... Of course Zeno knew perfectly well that Achilles would win a race with a tortoise, but [he] was drawing attention to the opposing theories on the nature of space and time. ... The Eleatic mathematical philosophers held that space and time are undivided wholes, or continua, that cannot be broken down into small indivisible parts. This was at variance with the Pythagorean idea that a line is made up of a series of points — like tiny beads or 'numerical atoms' — and that time likewise is composed of a series of discrete moments." [Burton, History of Mathematics: An Introduction 7th ed., p. 102.]

iv. Centering the compass point at the origin O, draw a (blue) circular arc AB passing through both legs of the angle so OA and OB have the same distance. Choose a sufficiently large radius and draw two (green) circles of that radius centered at A and B so they intersect at a point C inside the angle. Thus $AC = BC$. The line OC is the bisector of the angle. This is easily seen because the triangles $\Delta(AOC)$ and $\Delta(BOC)$ are congruent by SSS ($OA = OB$, $AC = BC$ and $OC = OC$.) Therefore the angles $\angle AOC$ and $\angle BOC$ are equal.



4. (a) *Use Pythagoras method to show* $\sqrt{5}$ *is irrational.*
For contradiction, assume that $\sqrt{5}$ is rational. That is

$$\sqrt{5} = \frac{p}{q}$$

where p and q are integers which we may assume to be in lowest terms: p and q have no common factors. Then

$$5q^2 = p^2.$$

This says that five divides p^2: $5|p^2$. But since 5 is prime it follows that $5|p$. This means $p = 5k$ for some integer k. Thus

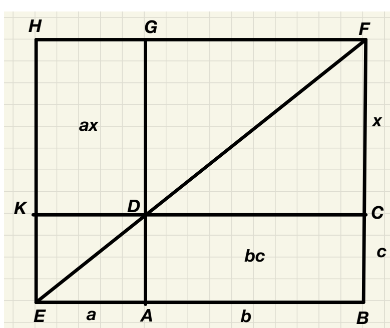
$$5q^2 = (5k)^2 = 25k^2$$

or

$$q^2 = 5k^2.$$

This says $5|q^2$. As before, since 5 is prime $5|q$. Thus we have reached a contradiction: both p and q have the common factor 5. Since the contrary statement is false, we conclude that $\sqrt{5}$ is irrational.

(b) *In Book II of Euclid's Elements, a geometric solution is found for the equation $ax = bc$ where a, b and c are positive constants. Mark off distances a and b on the line EB. Build rectangles $EADK$ and $ABCD$ of height c and extend the vertical lines EH to EK, AD to AG and BC to BF. Extend the diagonal ED until it crosses the line BF at F. Then the desired solution is the distance $x = CF$. Explain why $ax = bc$ from the following diagram.*



As given n the text, one way to see $ax = bc$ is by areas. Indeed the pairs of triangles $\triangle EBF \cong \triangle EHF$, $\triangle EAD \cong \triangle EKD$ and $\triangle DCF \cong \triangle DGF$ are congruent by SSS. Thus, subtracting areas

$$\begin{aligned} ax = \text{Area}(\square KDGH) &= \text{Area}(\triangle EHF) - \text{Area}(\triangle EKD) - \text{Area}(\triangle DGF) \\ &= \text{Area}(\triangle EBF) - \text{Area}(\triangle EAD) - \text{Area}(\triangle DCF) \\ &= \text{Area}(\square ABCD) = bc. \end{aligned}$$

Another argument involves similar triangles. Since EF is a common diagonal to the parallel lines EB and KC the angles $\angle AED = \angle CDF$ so that the right triangles are similar $\triangle EAD \sim \triangle DCF$. Thus

$$\frac{c}{a} = \frac{x}{b}$$

or

$$ax = bc.$$

5. (a) *Use the Euclidean Algorithm to find* $\gcd(686, 266)$. *(Other methods receive zero credit.)*

Run the Euclidian Algorithm for $\gcd(686, 266)$.

$$\begin{aligned}686 &= 2 \cdot 266 + 154\\ 266 &= 1 \cdot 154 + 112\\ 154 &= 1 \cdot 112 + 42\\ 112 &= 2 \cdot 42 + 28\\ 42 &= 1 \cdot 28 + 14\\ 28 &= 2 \cdot 14 + 0\end{aligned}$$

Thus $\gcd(686, 266) = 14$.

(b) *Prove that for all* n,

$$P(n): \quad \sum_{k=1}^{n} k(k+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof by induction. For the bas case $n = 1$,

$$\text{LHS.} = 1 \cdot 2 = 2, \qquad \text{RHS.} = \frac{1 \cdot (1+1)(1+2)}{3} = 2,$$

which are equal, so $P(1)$ holds. In the induction case, assume for some n that $P(n)$ holds (the induction hypothesis). Then for $n+1$, using the induction hypothesis,

$$\begin{aligned}\sum_{k=1}^{n+1} k(k+1) &= \left(\sum_{k=1}^{n} k(k+1)\right) + (n+1)(n+2)\\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)\\ &= (n+1)(n+2)\left(\frac{n}{3} + 1\right)\\ &= \frac{(n+1)(n+2)(n+3)}{3}\end{aligned}$$

which is $P(n+1)$. Since the base and induction cases hold, by induction $P(n)$ holds for all $n \in \mathbb{N}$.

The other method we have used to deduce the sum is the method of telescoping sums. One observes that for

$$f(n) = \frac{(n-1)n(n+1)}{3}$$

one has

$$\begin{aligned}f(k+1) - f(k) &= \frac{k(k+1)(k+2)}{3} - \frac{(k-1)k(k+1)}{3}\\ &= \frac{k(k+1)\big[(k+2) - (k-1)\big]}{3} = k(k+1).\end{aligned}$$

Thus

$$\begin{aligned}&1 \cdot 2 \quad + \quad 2 \cdot 3 \quad + \quad 3 \cdot 4 \quad + \quad \cdots \quad + \quad n(n+1) \quad =\\ &= [f(2) - f(1)] + [f(3) - f(2)] + [f(4) - f(3)] + \cdots + [f(n+1) - f(n)]\\ &= f(n+1) - f(1) = \frac{n(n+1)(n+2)}{3} - \frac{(0)1(2)}{3} = \frac{n(n+1)(n+2)}{3}.\end{aligned}$$